Risk-taking in financial networks

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Abstract
We analyze risk-taking behaviors of financial institutions linked through cross-shareholding relationships. We endogenize risk exposure accounting for default risk by modeling Value-at-Risk-based risk-management – that is targeted default probabilities – in the presence of extreme risk on asset return. We relate risk-taking behaviors to a centrality measure that captures the propagation of losses within the network, and show that network integration increases risk-taking levels and expected return. However, we show that network integration also results in larger expected shortfall, indicating greater exposure to losses for creditors. We explore the impact of the cross-shareholding network on the implementation of regulation, particularly through capital requirements, and identify key institutions, those with the highest influence on aggregate investments in risky assets. (JEL: C72; D85)

Keywords: Financial network; Risk-taking; Value-at-Risk.

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1. Introduction

Financial systems play a crucial role in fostering long-term economic growth by facilitating capital accumulation and efficient allocation of resources.[1] As intermediaries, financial institutions navigate investment opportunities that, albeit risky, promise substantial returns. However, their exposure to extreme risks—events of high magnitude but low frequency—can precipitate defaults with far-reaching consequences.[2] Such failures not only result in direct costs due to institutional defaults but also trigger widespread negative externalities, undermining confidence in the financial system at large. Given these stakes, institutions adopt risk management frameworks that balance risk-taking incentives with stability constraints, often shaping their strategic interactions within the financial network.

A common approach to managing firm-level risk exposure is Value-at-Risk-based Risk Management (VaR-RM), which has been extensively documented in literature.[3] Under this approach, firms strategically calibrate their risk exposure by setting a threshold on the maximum acceptable probability of failure. Yet, risk management does not occur in isolation: the exposure of one institution affects, and is affected by, the exposure of others within the financial system. Network effects introduce strategic considerations that shape risk-taking behavior. Diversification and risk-sharing mechanisms interact with these strategic incentives, making the financial network a crucial determinant of

1. See, for instance, Levine (2005) and, more recently, Petra Valickova and Horvath (2015).

2. These extreme risks are at the heart of actual financial regulation. For example, Solvency II, the directive that harmonizes European Union insurance regulation, calibrates prudential regulation on the notion of bicentenary events.

3. See for instance Dowd (1998) and Saunders (1999). Bodnar et al. (1998) document the use of VaR-RM practice in non-financial corporations in the US. Basak and Shapiro (2001) stress that VaR estimates are crucial not just as decision-making tools but also for controlling risk, aiming to keep market exposure within predefined levels.

equilibrium outcomes. In this context, understanding how strategic interactions influence risk-taking under VaR-RM is key to assessing financial stability. Despite its importance, the influence of financial networks on the risk-taking behavior of financial institutions through strategic interactions, particularly in managing acceptable risk levels amid extreme risks, has not been thoroughly explored. This paper fills this gap by examining how financial linkages affect institutional approaches to risk management.

In this study, we explore financial networks through the lens of cross-shareholding contracts, which are used for diversification, and have become increasingly prevalent among financial institutions.[4] According to regulatory data, global systemically important institutions hold comparable amounts of equity securities issued by other financial institutions relative to debt securities.[5] Both types of securities are used by the regulators to define regulatory buckets and the corresponding capital requirements. We develop a simple model that captures the essence of the strategic interaction among risk-taking behaviors conveyed by the shareholding network.

We model a financial system exposed to an extreme risk event – characterized by its high magnitude and low frequency – that poses a significant threat to individual institutions by potentially inflicting substantial losses on their risky assets. We study two scenarios regarding the systemic effect of shocks. In the first scenario, a shock to a given institution has no effect on the risky assets of other institutions. In the second scenario, the shock induces stress

4. Pollak and Guan (2017) argue that *"Between 2000 and 2015 the number of institutions with ownership in other institutions doubled in the United States. [...] Between 2011 and 2015 the total value of ownership of institutions by institutions increased by 211 percent"*.

5. Cf. Federal Financial Institutions Examination Council (FFIEC) or the European Banking Authority (EBA) database.

throughout the entire financial system, reducing all institutions' risky assets.[6] Within this framework, we assume that financial institutions manage this extreme risk based on Value-at-Risk-based Risk Management (VaR-RM). These strategies involve setting a level of risk-taking that aligns with a predetermined upper limit on the probability of default. This model allows to understand how the structure of financial networks influences the risk-taking behaviors of financial institutions through strategic interactions.

The model predicts that cross-shareholding networks create interdependencies among financial institutions' risk-taking decisions. When a shock hitting a financial institution stays localized and doesn't cause global stress to the financial system, the network acts as a form of mutual insurance in the face of a low-probability / high-impact risk. This combination of extreme risk and cross-shareholding fosters strategic complementarities in risk-taking. Conversely, when the shock creates a stress accross other institutions, the insurance mechanism is less effective, and risk-taking decisions become strategic substitutes.

We further explore these strategic interactions by introducing a centrality measure that captures an institution's position within the network and its influence on risk-taking behavior. This measure considers both the benefits of risk-sharing among well-connected institutions and the potential drawbacks of negative feedback loops from distressed entities. Our analysis indicates a nuanced relationship between network position and risk-taking. Still, more integrated cross-shareholding networks – characterized by higher cross-shareholdings levels – lead to increased risk-taking for all institutions when the shock is localized and, on average, when the shock stresses the financial system. Put another way, a more integrated network leads to higher expected return (for all institutions when the shock is localized and on average when the entire

6. To simplify our analysis and maintain tractability, we intentionally exclude the possibility of default contagion.

system is stressed) for a given default risk. This implies that the positive effects of increased integration through diversification outweigh the negative effects through feedback loops, enhancing the resilience of the institution affected by the shock and thereby encouraging increased risk-taking.

While networks can mitigate individual institutions' risk through diversification and support, we also show that they also magnify debt holders' exposition to default. By increasing risk-taking for a given default probability, the cross-shareholding network increases the loss given default, i.e. the expected shortfall. We demonstrate that, when accounting for adjustments in risk-taking, a more integrated network induces higher expected shortfall for all financial institutions when the shock is localized, and on average when it stresses the financial system. These results reveal a trade-off created by the cross-shareholding network between risk-taking and institutional default risk on one side, and debt-holder exposure to default on the other.

Finally, we show how, by adjusting liability-side balance sheet regulations, prudential regulation can mitigate excessive risk-taking while encouraging healthy levels of investment in risky assets. We also propose a 'key-player' policy approach on capital injection, pinpointing the specific institution where targeted capital injections could optimally increase aggregate risk-taking without impacting default probability, thereby enhancing the system's stability without compromising on necessary risk engagement.

Overall, our study sheds light on the intricate interplay between cross-shareholding networks, risk management strategies, and regulatory interventions, offering valuable insights into optimizing financial stability in the face of catastrophic risks.

Relationship to the literature. Our study adds to distinct, although in some respect complementary, strands of research. First, our paper contributes to the fast-growing literature on cross-shareholding networks (see Brioschi et al. (1989), Fedenia et al. (1994), or, more recently, Elliott et al. (2014)), and in particular it complements the line of work investigating endogenous risk-taking by financial institutions. Galeotti and Ghiglino (2021) analyze how cross-shareholding affects portfolio choices, in a model where firms choose to allocate wealth between a risk-free asset and a personal risky project, assuming away default. Focusing on risky projects with uncorrelated returns, risk-taking behaviors remain independent in their framework. Still, financial linkages alter investment decisions through diversification. In contrast, accounting for default generates strategic interactions in risk-taking decisions through a mutual insurance mechanism. In our context, the impact of financial networks on risk-taking decisions takes into account the interdependencies in risk-taking behaviors. Jackson and Pernoud (2019) incorporate both risk-taking and default, but limited to examples with binary (and independent or fully correlated) returns on the risky assets. Our paper is the first to study more generally the impact of cross-shareholding networks on risk-management, in a setup with possible institution defaults.

Our paper also adds to the literature on Value-at-Risk Risk Management (VaR-RM) by linking financial networks to VaR-RM. The value-at-risk was originally introduced by Markowitz (1952) and Roy (1952) in an attempt to optimize profit so as to incorporate the risk of high losses. In its current form, VaR was presented in 1989 by JP Morgan in their risk management tool called the RiskMetrics, and is used in banking regulation. With a view to accounting for debt-holders exposure to default, recent extension of the VaR approach include the C-VAR (C for conditional); See Rockafellar and Uryasev (2000), Rockafellar and Uryasev (2002).[7] We show how VaR-RM can impact expected shortfall when financial institutions are linked through cross-shareholding, and in particular that denser networks leads to larger expected shortfall (for all institutions under no stress, in average under stress).

7. As pointed out by Jorion (1998), VaR-RM can also underestimate risk due to its reliance on short-term history and risk concentration.

This paper also adds to the literature on prudential regulation. Prudential regulation through cash or capital requirements has been shown to be a useful and powerful tool to deal with excessive risk-taking by institutions and to reduce default risk (Hellmann et al., 2000; Decamps et al., 2004).[8] Implemented by financial regulators since the early 1990s (through the 1988 Basel Accord, also known as Basel I), such regulation gained in complexity thereafter to account for specific risks (e.g., market risk, liquidity risk, and operational risk). It dampens solvency risk without the social cost of bailouts, or their effects induced through moral hazard when anticipated (Freixas and Rochet, 2013). However, during the 2007 financial crisis, prudential regulation proved insufficient to limit excessive risk-taking, notably because of the extent of financial linkages – see, for example, the cases of Lehman Brothers and American International Group discussed in Glasserman and Young (2016). Current regulation treats financial linkages, whether in the form of equity or debt holdings, in the same way. However, we show that cross-shareholding networks can positively influence risk-taking without increasing the likelihood of default. There is also a nascent literature on public intervention in financial networks. Elliott et al. (2014) study the effect of reallocations of cross-holdings that leave the market value of institutions unchanged and find that they are not effective in avoiding the first failure. Leduc and Thurner (2017) study the effect of transaction-specific taxes when institutions are connected through debt contracts and subject to liquidity shocks and show that this can reduce contagion. Finally, Demange (2018) and Jackson and Pernoud (2019) discuss the optimal ex-post intervention, through bailouts or cash injection. We complement these literatures by analyzing a prudential policy consisting in a capital injection intervention taking into account the interdependent risk-taking behaviors of financial institutions.

8. Cash requirements correspond to constraints on the asset side, whereas capital requirements affect the liability side.

We conclude by discussing the recent literature on contagion, namely, the spread of shocks between linked institutions.[9] Although we do not incorporate contagion, our model has close connections to some papers in that literature. The structure of risk we consider, with one large negative shock hurting one institution at a time, is similar to Cabrales et al. (2017), who model financial linkages as investments by institutions in each other's projects and analyze the optimal network structure depending on projects' riskiness. A comparative statics on integration is also discussed in Elliott et al. (2014), who consider additional frictions through default costs in a model of linear cross-holdings.[10] In all of the above papers, the initial risk faced by each institution is exogenous, while our paper rather models shocks endogenous to investment choices by institutions. A recent exception is Shu (2024), who models unsecured inter-institution debt contracts, mostly on regular networks, and obtains complementarities in risk-taking behaviors. The mechanism generating complementarities in risk-taking behaviors is different from ours. Shu (2024) models an inter-debt network in which limited liability (referred to as cross-subsidies in Shu (2024)) gives incentives to an institution to increase its level of risk-taking when an institution it is linked to defaults. As banks default probability increases with risk-taking, this creates a strategic complementarity. A distinct mechanism applies to our paper. Modeling a cross-shareholding network rather than a debt network, the mechanism generating strategic complementarities relies on the insurance device that the network brings to a firm hit by a shock. Moreover, our work highlights both positive

9. The effect of financial networks on contagion, when institutions are linked through debt contracts is analyzed in Allen and Gale (2000), Acemoglu et al. (2015), Glasserman and Young (2016), Acemoglu et al. (2015), and Glasserman and Young (2016). For a recent survey on the transmission of liquidity shocks in large networks, see Gai and Kapadia (2019).

10. Although they model links as shareholding, Elliott et al. (2014) view them as "debt contracts around and below organizations' failure thresholds" and assume that default costs spread in the network.

and negative aspects of cross-shareholding networks with respect to risk-taking and the challenge they poses for public policies. It is particularly noteworthy that the negative effects of financial networks, in our model, are independent from any contagion considerations.

The remainder of this paper is organized as follows. The model is presented in Section 2. In Section 3, we characterize the optimal levels of risk-taking under VaR-risk management, we undertake a comparative statics on integration, and we analyze simple network structures. Section 4 examines the situation in which the shock induces a stress to all financial institutions. Section 5 explores the impact of VaR-risk management on expected shorfall. Cash injection policy is analyzed in Section 6. We conclude in Section 7. All proofs are relegated to Appendix A. Appendix B examines risk-taking decisions in the absence of VaR-RM. Appendix C explores the case where several firms suffer the shock at once, and Appendix D analyzes directed cross-shareholding networks.

2. The model

We consider a network of $n \geq 2$ financial institutions potentially linked through cross-shareholding. These institutions can be, for example, banks, insurance companies or pension funds. We consider a two-period model in which every institution is liquidated after asset return realization. This simple model allows to capture the effects of cross-shareholding.

We introduce the following notation. Matrices are written in block and bold letters, and vectors in lower case and bold letters; the superscript ^{T} stands for the transpose operator. Numbers and entries of matrices are in lower case. We let $\mathbf{I}$ be the identity matrix of order n; $\mathbf{0}$ and $\mathbf{1}$ represent the vectors of zeros and ones of dimension n, respectively.

The financial network. At $t = 0$, each financial institution $i \in \mathcal{I} = \{1, 2, \cdots, n\}$ is financed by debt (or deposit) d_i, by equity held by outside investors e_i, and by equity held by other financial institutions in the network.[11] We let matrix $\mathbf{P} = (p_{ij})_{i,j \in \mathcal{I}^2}$ represent the cross-equity holding network, where p_{ij} represents the amount held by institution i in institution j. Throughout the paper, the cross-equity holding matrix is exogenous, reflecting strategic long run perspective.[12] To isolate the role of the cross-shareholding network on the level of risk taken by each institution, we focus on comparable institutions in two ways. First, we focus on homogeneous external equities and debt, i.e. $e_i = e, d_i = d$ for all i. Our whole analysis is straightforwardly generalized to heterogeneous debt d_i and external equity e_i. We relax this homogeneity assumption in Section 6 on cash injection. Second, to neutralize resource effects (that is the fact that some institutions may have larger amounts to invest in financial assets than others), we assume $\mathbf{P}^T\mathbf{1} = \mathbf{P}\mathbf{1} = (p_i)_{i \in \mathcal{I}}$, meaning that for each financial institution, the sum of incoming amounts held by other institutions is equal to the sum of amount made in other institutions. We relax this assumption in Appendix D and show that our main insights remain valid qualitatively. By the assumption $\mathbf{P}^T\mathbf{1} = \mathbf{P}$, the resource of each financial institution is equal to $e + d$. That resource is allocated between investment in a risk-free asset (with normalized return equal to 1), $x_i \geq 0$, an investment in a institution-specific risky asset, $z_i \geq 0$. In the following, we focus on the effect of the cross-shareholding network on portfolio allocation (x_i, z_i).

The balance sheet of institution i at $t = 0$ (i.e., before realization of risk) can then be represented as in Fig. 1 (left panel). This leads to the following accounting equation at $t = 0$ (taking into account that $\sum_{j \in \mathcal{I}} p_{ij} = \sum_{j \in \mathcal{I}} p_{ji}$):

$$x_i + z_i = e + d \tag{1}$$

11. We disregard debt contracts among financial institutions.

12. Current regulatory requirements, which are based on annual reports of cross-shareholdings, confirm a long-term perspective.

Assets	Liabilities
x_i	d
z_i	e
$\sum_j p_{ij}$	$\sum_j p_{ji}$

Assets	Liabilities
x_i	ρd_i
$\mu_i z_i$	
$\sum_j a_{ji} v_j$	v_i

FIGURE 1. Balance sheet of financial institution i. Left panel: at $t = 0$. Right panel: at $t = 1$.

Letting $\mathbf{z} = (z_i)_{i \in \mathcal{I}}$ represent the profile of investments in risky assets, we have $\mathbf{z} \in [\mathbf{0}, \mathbf{e} + \mathbf{d}]$ from the balance sheet equation (1). At $t = 1$, returns on investments are realized, institutions liquidated, and their values (if any) are distributed among their shareholders. Let v_i be the total equity value of institution i. We denote by $a_{ij} = \frac{p_{ij}}{e + \sum_k p_{kj}}$ the share of the equity of institution j held by institution i, that is, as $p_j = \sum_k p_{kj}$, $a_{ij} = \frac{p_{ij}}{e + p_j}$. We define the corresponding matrix of shares $\mathbf{A} = (a_{ij})_{(i,j) \in \mathcal{I}^2}$. We let $\rho \geq 1$ represent the deterministic (gross) cost on debt or deposit, and let $\tilde{\boldsymbol{\mu}} = (\tilde{\mu}_i)_{i \in \mathcal{I}}$ be the vector of stochastic return on the risky asset of each institution. Figure 1 (right panel) presents the balance sheet of institution i for a given realization μ_i of the risky asset at $t = 1$.

In the event that $x_i + \mu_i z_i - \rho d + \sum_{j \neq i} a_{ij} v_j < 0$, firm i defaults: all assets go to debt repayment and equity-holders get no value. Because of limited liability of equity holders (they are not liable for losses), the total equity value of institution i is $v_i = \max(x_i + \mu_i z_i - \rho d + \sum_{j \neq i} a_{ij} v_j, 0)$.

Using the accounting equation (1) at $t = 0$, and considering the stochastic nature of risky asset returns, the accounting equation for every surviving institution i at $t = 1$ becomes

$$\tilde{v}_i(\tilde{\boldsymbol{\mu}}, \mathbf{z}) = \max\Big((\tilde{\mu}_i - 1) z_i + e - (\rho - 1) d + \sum_{j \neq i} a_{ij} \tilde{v}_j(\tilde{\boldsymbol{\mu}}, \mathbf{z}), 0\Big) \tag{2}$$

ASSUMPTION 1. $e - (\rho - 1) d > 0$

Under Assumption 1, the initial level of equity ensures that a financial institution remains solvent when it does not invest in risky assets (i.e., $v_i > 0$ when $z_i = 0$).

The structure of risk. We focus on extreme and rare events, likely to put one institution into financial distress. We therefore assume that only one institution at a time can be hurt by this large negative shock. When not hurt, the return on a bank risk asset is $r > 1$. However, with probability q_0, a large negative shock hits a single institution at random (with uniform probability). The return on risky asset for the bank hit by the shock suffers a stochastic loss $\tilde{s}$, distributed on the non-negative support $[s_0, +\infty)$, $s_0 > r - 1$ (leading to $\mu_i < 1$), with cumulative function H and average value $\overline{s}$.

Formally, we assume that for every bank i,

$$\tilde{\mu}_i = \begin{cases} r & \text{with probability } 1 - \frac{q_0}{n} \\ r - \tilde{s} & \text{with probability } \frac{q_0}{n} \end{cases} \tag{3}$$

We assume for now that the shock hitting one bank does not strongly deteriorate the health of the financial system. We examine in Section 4 a more general setup in which, once a firm is hit by the shock, other institutions also suffer a negative shock on their returns.

Assumption 2. *$\mathbb{E}(\tilde{\mu}_i) > 1$ $\forall i$.*

Assumption 2 implies that investment in the risky assets is still worthwhile, implying in particular that the expected return on one institution's investment, which writes $\mathbb{E}(\mu_i) \cdot z_i + x_i$, increases with its level of risk-taking z_i.

Value-at-Risk Management. In this setup, the institution's decision reduces to allocating its resources $e + d$ between the risk free asset and its specific risky asset. This optimal portfolio management by financial institutions is assumed to follow a Value-at-Risk Management principle. That is, each financial institution maximizes its expected equity value $\mathbb{E}(\tilde{v}_i)$,[13] under the constraint of complying to a maximum acceptable probability of default.[14] To isolate pure network effects, the acceptable default probability is assumed to be homogeneous across institutions. We denote this common maximal acceptable probability of default by β (e.g., that value can be set by the regulator). The model can easily be extended to heterogeneous values β_i, to account for individual characteristics, such as institution size. Each institution i then solves:

$$\max_{z_i \in [0, e_i+d_i]} \quad \mathbb{E}\Big(\tilde{v}_i(\tilde{\boldsymbol{\mu}}, \mathbf{z})\Big) \tag{4}$$

$$\text{s.t.} \qquad \mathbb{P}(\tilde{v}_i(\tilde{\boldsymbol{\mu}}, \mathbf{z}) = 0) \leq \beta$$

Note that $\tilde{v}_i(\tilde{\boldsymbol{\mu}}, \mathbf{z}) = 0$ when $(\tilde{\mu}_i - 1)z_i + e - (\rho - 1)d + \sum_{j \neq i} a_{ij}\tilde{v}_j(\tilde{\boldsymbol{\mu}}, \mathbf{z}) \leq 0$.[15] Focusing on environments in which the managerial constraint is binding for all institutions (that is in which β is low enough),[16] we study in the following

13. Managers' and equity-holders' objectives are assumed to be aligned. We thus ignore agency issues inside the institution.

14. The Value-at-Risk of financial institutions is taken into account by financial regulators; e.g., in Basel III and Solvency II. The value at risk is defined by the Basel Committee on Banking Supervision as "A measure of the worst expected loss on a portfolio of instruments resulting from market movements over a given time horizon and a pre-defined confidence level" (BCBS, 2019).

15. Our results are insensitive to institutions considering their external equity value (referred to as *market value* by Elliott et al. (2014)): $v_i \cdot e / \left(e + \sum_{k \in \mathcal{I}} p_{ki}\right)$, which is proportional to their total equity value.

16. The upper bound on β for which the constraint is binding depends on the cross-shareholding network. For any institution i, β has to be lower than the probability that institution i has a negative value when receiving the shock and given risk-taking levels are set at their upper bounds; I.e., $\beta < \frac{q_0}{n}\mathbb{P}\Big(s > \alpha_i\Big)$, where $\alpha_i = \frac{1}{m_{ii}}\sum_k m_{ik}((r-1)(e+d) + e - (\rho-1)d)$ with $\mathbf{M} = (\mathbf{I} - \mathbf{A})^{-1}$; or, letting H be the cumulative distribution of the shock s, $\beta < \frac{q_0}{n}(1 - H(\alpha_i))$. A sufficient condition on the $\max_i \alpha_i$ follows directly.

risk-taking levels such that:

$$\mathbb{P}(\tilde{v}_i(\tilde{\boldsymbol{\mu}}, \mathbf{z}) = 0) = \beta \quad \forall i \tag{5}$$

As is clear from the above program, the risk-taking level chosen by each institution depends on the entire cross-shareholding network $\mathbf{A}$.

Remark 1. Appendix B provides an upper bound on the probability of occurrence of the adverse event for which, when not constrained by value-at-risk risk management, institutions take the maximum amount of risk.

3. Risk-taking under Value-at-Risk Management

In this section, we solve the system of optimal risk-taking under VaR-RM in a context where the shock hitting a financial institution does not imply a stress on the financial system, and we undertake a comparative statics analysis with respect to the level of integration of the cross-shareholding network.

3.1. Characterization

We describe how VaR-RM shapes institutions' risk-taking. As the negative shock only hits one institution at time, under Assumption 1, a financial institution can only default in cases it is hit by the large negative shock on its asset; and in that case, the values of the other institutions are necessarily positive (as $r > 1$). We highlight that in these circumstances, the financial network favors risk-taking.

We define matrix $\mathbf{M} = (\mathbf{I} - \mathbf{A})^{-1}$, and matrix $\mathbf{C}$ with null diagonal ($c_{ii} = 0$) and off-diagonal entry $c_{ij} = \frac{m_{ij}}{m_{ii}}$. The Bonacich centrality of the cross-shareholding matrix, $\mathbf{b} = (b_i)_{i \in \mathcal{I}} = (\mathbf{I} - \mathbf{A})^{-1}\mathbf{1}$, will prove key in our analysis. It measures the influence of each institution on another by aggregating all the paths linking the two, where paths are weighted by the product of involved shares. This centrality measure takes a simple form here:

Observation 1. *For all* $\mathbf{P}$ *such that* $\mathbf{P}^T\mathbf{1} = \mathbf{P}\mathbf{1} = (p_i)_{i\in\mathcal{I}}$*, and all* e*, we have* $b_i = 1 + \frac{p_i}{e}$.

We pursue with the characterization of risk-taking behaviors. Assumption 1, together with $r > 1$, guarantees that v_i is positive when institution i is not hit by the large negative shock. Default of institution i may occur only if it suffers the shock. Institution i's value-at-risk constraint then corresponds to the probability of not surviving to the adverse shock being equal to β. Formally the VaR-constraint of institution i writes:

$$\frac{q_0}{n}.\mathbb{P}\left(\underbrace{(r-\tilde{s}-1)z_i + e - (\rho-1)d + \sum_{j\neq i} c_{ij}\left[(r-1)z_j + e - (\rho-1)d\right]}_{\tilde{v}_i \ | \ \mu_i = r-\tilde{s}, \ \mu_j = r \ \forall j\neq i} \leq 0\right) = \beta \quad (6)$$

Define $t_{1-\frac{n\beta}{q_0}}$ as the $(1-\frac{n\beta}{q_0})$th quantile of the distribution of $\tilde{s}$ and $\ell_\beta = r - t_{1-\frac{n\beta}{q_0}}$; ℓ_β can then be understood as the value-at-risk at level $(1-\beta)$ of a unit investment in risky asset (see footnote 14). A low value of ℓ_β then reflects tight risk-management (a low value of β) or large market risk (a distribution of $\tilde{s}$ with heavy right tail). The equilibrium is then defined by

$$(\ell_\beta - 1)z_i + e - (\rho-1)d + \sum_{j\neq i} c_{ij}\left[(r-1)z_j + e - (\rho-1)d\right] = 0 \ \forall i \quad (7)$$

To address interesting cases, we assume that the value-at-risk of each institution, ℓ_β, is bounded from above:

Assumption 3. $\ell_\beta < 1$.

Assumption 3 corresponds to tight risk-management as it leads to a sufficiently low value of the maximum acceptable default probability β.[17] Note

17. In the current regulation, β is set to 1% in the financial sector (Basel II) and 0.5% in the insurance industry (Solvency II).

that, when $\ell_\beta > 1$ the VaR-constraint is never binding: the default probability is lower than β whatever $z_i \in [0, e+d]$ and institutions put all their resource to the risky asset. Let $\varepsilon_\beta = \frac{r-1}{1-\ell_\beta}$, which is positive under Assumption 3. We obtain that optimal risk-taking levels solve, for all i:

$$z_i - \varepsilon_\beta \sum_{j \neq i} c_{ij} z_j = \frac{e-(\rho-1)d}{1-\ell_\beta} \frac{b_i}{m_{ii}} \tag{8}$$

and, as $\varepsilon_\beta > 0$, the risk-taking levels under VaR-RM are strategic complements. This pattern of strategic complementarities stems from the fact that as one institution suffers a negative shock, the other institutions in the network always provide support to institution i through cross-shareholding links. Since the value received by institution i through its shares in the financial system is increasing in other institutions' investment in their risky assets (as $r > 1$), the higher this investment, the more institution i receives in case of shock. Institution i's investment in its risky asset can then be higher while maintaining the same default probability. We observe that the risk-taking level under VaR-RM for an isolated institution is $z_i^* = \frac{e-(\rho-1)d}{1-\ell_\beta}$, which is positive under Assumptions 1 and 3.

The complementarities of the interactions, together with the upper bounds on values of z_i, guarantee the existence of a solution $\mathbf{z}^*$ to the system of programs (2) $\forall i$. Assumptions 2 to 3 guarantee that the solution is unique and positive (see Belhaj et al., 2014). Some levels of risk-taking can still reach the upper bound $e+d$. Now, considering the system of best-responses functions BR, with $BR_i(\mathbf{z}) = \varepsilon_\beta \sum_{j \neq i} c_{ij} z_j + \frac{e-(\rho-1)d}{1-\ell_\beta} \frac{e+p_i}{e\ m_{ii}}$ $\forall i$, any interior solution satisfies $\mathbf{z}^* = BR(\mathbf{z}^*)$. We can then focus on interior solutions through the following assumption.

Assumption 4. $BR((e+d)\mathbf{1}) < (e+d)\mathbf{1}$.

Through complementarity in risk-taking levels, Assumption 4 guaranties that the VaR-RM imposes a binding constraint for all institutions. The risk-taking behaviors are then given by the following theorem:

THEOREM 1. *Under Assumptions 1 to 4, there is a unique solution to the set of programs (2). This solution is interior, risk-taking levels are strategic complements and given by*

$$\mathbf{z}^* = \frac{e-(\rho-1)d}{r-1}\Big[(1+\varepsilon_\beta)(\mathbf{I}-\varepsilon_\beta\mathbf{C})^{-1}\mathbf{1}-\mathbf{1}\Big] \tag{9}$$

The interior solution $\mathbf{z}^*$ builds on a centrality measure, $(\mathbf{I}-\varepsilon_\beta\mathbf{C})^{-1}\mathbf{1}$, that expresses institutions' risk-taking levels as a function of their position in the (weighted) network of cross-shareholding $\mathbf{A}$ – recall that $c_{ij} = \frac{m_{ij}}{m_{ii}}$ where $\mathbf{M} = (\mathbf{I}-\mathbf{A})^{-1}$. This centrality echoes Bonacich centrality presented in Observation 1 (as $\mathbf{b} = \mathbf{M1}$), but self-loops also play a role in relation with the adverse event. On the one hand, central institutions benefit from other institutions' values in case of shock, which tends to increase their risk-taking level; on the other hand, following a shock, the network also amplifies the loss in value of the shocked institution through feedback effects. This is why self-loops play a role: if a given firm is hit by a shock, the value of those institutions with cross-investments in it decrease, which tends to decrease its own value. The centrality measure presented in Theorem 1 captures these complex networked interactions between the shocked institution and the other institutions.

To illustrate the distinction between Bonacich centrality, that drives institutions' values, and the centrality that drives risk-takings under VaR-RM, consider the network depicted in Figure 2. In that example, we let $\mathbf{G} = (g_{ij})$, with $g_{ij} \in \{0, 1\}$, represent the binary network supporting cross-holding in equities: we set $p_{ij} = p \cdot g_{ij}$ and assume $\mathbf{G}^T = \mathbf{G}$. Denoting by $\delta_i = \sum_j g_{ij}$ the degree of institution i, we get $p_i = p\delta_i$. The degree of institution 3, and thus its Bonacich centrality (by Observation 1), is larger than that of institution 4, but its risk-taking level is lower.

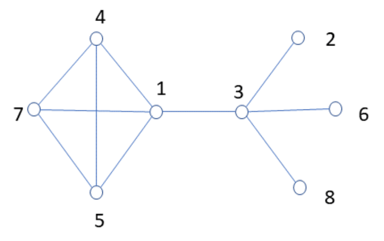

FIGURE 2. Bonacich centrality and risk-taking. We set $p_{ij} = p \cdot g_{ij}$ and $\mathbf{G}^T = \mathbf{G}$. For $\rho = 1.4$, $e = 200$, $d = 200$, $p = 20$, $\ell_\beta = 0.6$, $r = 1.7$, $\mathbf{z}^* \simeq (887, 445, 761, 765, 765, 445, 765, 445)$. Here z_3 is lower than z_4 whereas the Bonacich of institution 3, which is aligned with its degree, is larger.

This tradeoff is even more salient under tight risk-management. We obtain the following corollary:

COROLLARY 1. *When risk-management is tight ($\varepsilon_\beta = \frac{r-1}{1-\ell_\beta} \to 0$), the risk-taking level of institution i: z_i^* is proportional to the ratio $\frac{b_i}{m_{ii}} = \frac{e+p_i}{e\ m_{ii}}$.*

Bonacich centrality aggregates the share of other institutions' values held by one institution ($b_i = \sum_j m_{ij}$ and $\mathbf{M} = \sum_{q=0}^{\infty} \mathbf{A}^q$). Institutions with higher Bonacich centrality receive more from others through the cross-shareholding network and can therefore take more risk for a given default probability. Now, the network may also make an institution more exposed to its own value, and therefore to its own level of risk-taking, through self-loop (m_{ii}). Institutions with higher self-loop centrality then suffer more from a shock on their risky asset and can therefore take less risk (for a given probability of default). Corollary

1 states that under tight risk-management (i.e., a large value-at-risk ℓ_β in absolute term: $\varepsilon_\beta \to 0$ when $\ell_\beta \to -\infty$), risk-taking decisions result from a trade-off between these two effects.

We also observe that, on regular networks, for which Bonacich centralities are homogeneous, risk-taking levels can be differentiated, when self-loop centralities differ. Furthermore, the ratio $\frac{b_i}{m_{ii}}$ that stems from the cross-shareholding network is not necessarily favorable to more central institutions. The next example depicted in Figure 3 illustrates that the ordinal ranking of this ratio can differ from the ranking of degrees; in this example, we set $p_{ij} = p \cdot g_{ij}$ and assume $\mathbf{G}^T = \mathbf{G}$.

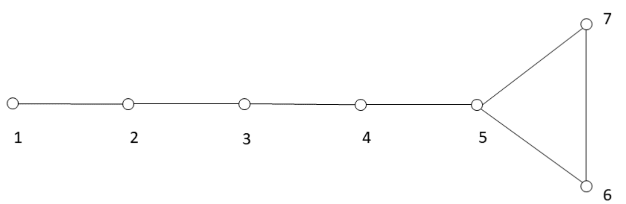

Figure 3. In this network, degree and ratio $\frac{b_i}{m_{ii}}$ are not aligned. We set $p_{ij} = p \cdot g_{ij}$ and $\mathbf{G}^T = \mathbf{G}$. Here, parameters $e = p = 20$ have been used to generate the cross-shareholding matrix. The profile of ratio is $(3.62, 4.45, 5.02, 5.26, 5.16, 4.51, 4.51)$. Whereas institution 5's degree is larger than that of institution 4, we have $\frac{b_5}{m_{55}} < \frac{b_4}{m_{44}}$.

Among all network structures, core-periphery networks[18] have been shown to be reasonably representative of real financial networks, both in terms of inter-institution lending; see, e.g.,Craig and von Peter (2014) and cross-shareholding Rotundo and D'Arcangelis (2014). The simplest case is perhaps that of an undirected star network (i.e., the core is reduced to a single institution). We find:

PROPOSITION 1. *Consider Assumptions 1 to 4. In an undirected star network, the risk-taking level of the central institution is higher than that of the peripheral institutions.*

The proof of Proposition 1 rests on the specification of (asymmetric) matrix **A**. We first prove that the ratio $\frac{b_i}{m_{ii}}$ for the center is greater than in any peripheral institution.[19] By Corollary 1, this statement proves the result for sufficiently large negative shocks. We then extend the proof to arbitrary values of ε_β by using the ranking of centralities in an argument by induction.

REMARK 2 (*Multiple shocks*). Allowing for more than one shock makes the network less useful to the institutions suffering the shocks, thereby leading to more restrictions on risk-taking. See Appendix C for more details.

3.2. Comparative statics on network integration

By network integration we mean an increase in the matrix of shares **A**. Such integration can arise for various reasons. For instance, an increase in the

18. Core-periphery networks are networks in which highly interconnected nodes – called the core – coexist with nodes loosely connected (both to the core and among themselves) – called the periphery.

19. The ratio of Bonacich centrality over self-loop centrality is not necessarily favorable to central institutions in linear-in-sum models where a sum of row can exceed unity. For instance, in a star network, this ratio can be favorable to peripheral agents under sufficiently high values of interaction.

matrix of cross-equities $\mathbf{P}$ leads to an increased matrix of shares $\mathbf{A}$ when the external equity e is sufficiently large.[20] Similarly, when equity investments are homogeneous across financial institutions (i.e., $p_{ij} \in \{0, p\}$), an increase of the amount p invested in other institutions entails greater integration.

How does more integration - an increase in the cross-shareholding matrix $\mathbf{A}$ - affect risk-taking under VaR-RM? Increasing shareholding has an ambiguous effect a priori: (i) it propagates the negative shock on one institution's asset to the whole network, but (ii) it propagates the (necessarily positive) value of other institutions to the institution hit by the negative shock. We find:

PROPOSITION 2. *Consider Assumptions 1 to 4. An increase of the cross-shareholding network* $\mathbf{A}$ *increases optimal risk-taking for all institutions.*

By Proposition 2, the adverse increased feedback effects are always dominated by the positive increased complementarities helping institutions to survive to the bad shock. One implication of this proposition is that every nonempty shareholding network leads to high levels of risk-taking, and therefore higher expected returns, for every institution with regard to the no-shareholding case (i.e., the empty network).

REMARK 3 (*Directed shareholding network*). Directed shareholding networks (that is, relaxing the assumption $\mathbf{P}^T\mathbf{1} = \mathbf{P}\mathbf{1}$) bring a resource effect in the accounting equation of the institutions' balance sheet: institutions with more investors benefit from higher resource. Taking into account this resource effect adds a term in the risk-taking levels under VaR-RM $\mathbf{z}^*$. Some of our results are kept unchanged: central institutions in directed stars take more risk under VaR-RM, and simulations over a large number of networks generated by popular

20. Assume $\mathbf{P}' > \mathbf{P}$. Then, $a'_{ij} > a_{ij}$ if and only if $e > \bar{e}_{ij} = \frac{p_{ij}P'_j - p'_{ij}P_j}{p'_{ij} - p_{ij}}$. Taking $e > \max_{(i,j)} \bar{e}_{ij}$ guarantees $\mathbf{A}' > \mathbf{A}$.

random network generation models confirm that more integration fosters risk-taking in general (see Appendix D for more details).

4. A shock stressing the financial system

In the benchmark model presented above, the shock hitting an institution does not lower others' asset returns. Now assume that, once a firm is hit by the shock, the system enters into stress in the sense that the other institutions also experience bad returns (indirect economic mechanisms, as fire-sale, can explain this phenomenon). We show in the section that when the stress is large – leading to negative returns for all institutions – the nature of strategic interactions is qualitatively affected, and risk-taking decisions become strategic substitutes. However, the analysis reveals that the network still enhances risk-taking levels and expected returns with respect to the no-network case.

We model a stressed financial system as follows. With probability $1 - q_0$, the system is not stressed, and the return on every institution's risky asset is $r > 1$. With probability q_0, the return on the risky investment of every institutions falls to $\underline{r} < r$, and a large negative shock hits a single institution at random with uniform probability. The institution hit by the shock still suffers a stochastic loss $\tilde{s}$, now distributed on the non-negative support $[s'_0, +\infty)$, $s'_0 > \underline{r} - 1$ (leading to $\mu_i < 1$), with cumulative function H' and average value $\overline{s'}$.[21] Formally, we assume that for every institution i,

$$\tilde{\mu}_i = \begin{cases} r & \text{with probability } 1 - q_0 \\ \underline{r} & \text{with probability } \frac{n-1}{n} q_0 \\ \underline{r} - \tilde{s} & \text{with probability } \frac{q_0}{n} \end{cases} \tag{10}$$

21. This structure of risks echoes that of Cabrales et al. (2017), who model rare and large shocks on gross return through a deterministic return with fixed probability and two alternatives with either a small or a large shock.

Figure 4 illustrates the structure of the stochastic return on institution i's risky asset.

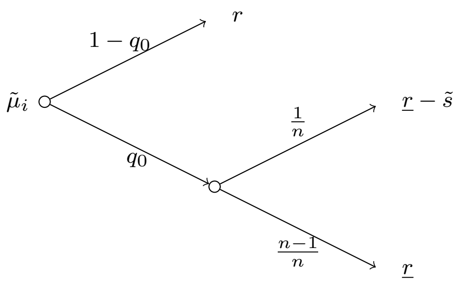

FIGURE 4. The stochastic return $\tilde{\mu}_i$ of institution i

We assume that, in spite of the stress, institutions only default in states of nature when they are hit by the negative shock. A sufficient condition for this to hold is that, for all i, $m_{ii}z_i^* < \min_{j \neq i} m_{ij}z_j^*$. Put another way, we restrict to settings in which each institution is more exposed to its own risk that to others. Risk taking levels under VaR-RM then still satisfies equation (8), where now $\varepsilon_\beta = \frac{\underline{r}-1}{1-\ell_\beta}$. The magnitude of the stressed environment then proves key to understand risk-taking interactions. Recalling that $\ell_\beta < 1$, equation (8) implies:

PROPOSITION 3. *Consider Assumptions 1 to 4. When $\underline{r} > 1$ (or $\varepsilon_\beta > 0$), risk-taking levels are strategic complements. When $\underline{r} = 1$ (or $\varepsilon_\beta = 0$), risk-taking levels are independent. When $\underline{r} < 1$ (or $\varepsilon_\beta < 0$), risk-taking levels are strategic substitutes.*

Equilibrium multiplicity can be an issue with strategic substitutes in general.[22] Yet, in our model uniqueness still holds for any $\underline{r} \geq \ell_\beta$ (or $\varepsilon_\beta \geq -1$); the proof is immediate from Proposition 6 thereafter; indeed, multiplicity requires corners, which do not emerge here. Still, the nature of the interaction affects the comparative statics on network integration (i.e. Proposition 2). A

22. See Bramoullé et al. (2014) for sufficient conditions in network games.

higher network integration does not necessarily increase the optimal level of risk-taking of each institution when $\underline{r} < 1$. However, even in this case, this is true on average:

Proposition 4. *Consider Assumptions 1 to 4. An increase in the shareholding matrix that keeps the risk-taking levels interior entails increased average risk-taking on any network.*

Therefore, whatever the value or $\underline{r}$, network integration increases the average expected return.

We examine now the impact of an increase of the default probability on average risk-taking:

Proposition 5. *Consider Assumptions 1 to 4. An increase in the default probability β that keeps the risk-taking levels interior entails increased average risk-taking on any network.*

Therefore, even in the presence of a shock stressing the system, the cross-shareholding network entails higher expected return and VaR-RM can be understood as a trade-off between expected return and institution default probability.

Table 5 illustrates the role of networks by comparing the average expected return for various network structures when the shock is Pareto distributed or exponentially distributed. Given the structure of risk, the expected return is equal to $(1-q_0)r + \frac{n-1}{n}q_0\underline{r} + \frac{q_0}{n}(\underline{r} - \mathbb{E}(\tilde{s}))$.[23] We fix parameter values such that $\lambda = \frac{a-1}{s_0}$, so that both probability distributions share the same mean (the mean of the Pareto (resp. exponential) distribution is $\frac{as_0}{a-1}$ (resp. $s_0 + \frac{1}{\lambda}$). Table

23. The Pareto distribution is such that $h_a(s) = \frac{as_0^a}{s^{a+1}}$ over the interval $[s_0, +\infty)$, for $a > 1$; the exponential distribution satisfies $h_\lambda(s) = \lambda e^{-\lambda(s-s_0)}$ over the interval $[s_0, +\infty)$, for $\lambda > 0$. For the Pareto law, $\mathbb{E}(\tilde{s}) = \frac{a}{a-1}s_0$, while for the exponential law, $\mathbb{E}(\tilde{s}) = s_0 + \frac{1}{\lambda}$.

	Star	Wheel	two-Core	Complete
Power law	210.54	210.56	210.64	210.79
Exponential	210.87	210.90	210.97	211.13

FIGURE 5. Average expected return of the financial system for different networks and different probability distributions of the shock with same mean; $n = 6, \rho = 1.1, d = 100, e = 100, r = 1.2, \underline{r} = 0.7, p = 1, q_0 = 0.1, \beta = 0.1, s_0 = 1, a = 10, \lambda = \frac{a-1}{s_0} = 9$.

5 shows that both network and nature of the shock have a strong impact on the average risk-taking.

Moreover, we can show that the cross-shareholding network still increases the risk-taking level of each institution as compared to the no-network case:

PROPOSITION 6. *Consider Assumptions 1 to 4. Consider any non-empty and undirected cross-shareholding network. When $\underline{r} > \ell_\beta$ (or $\varepsilon_\beta > -1$), the network favors risk-taking with respect to isolated institutions. When $\underline{r} = \ell_\beta$ (or $\varepsilon_\beta = -1$), risk-taking levels are identical to those taken in isolation.*

Proposition 6 highlights that, even when the entire financial system is stressed, the existence of cross-shareholding linkages entails an increase in risk-taking levels of every institutions. Indeed, even if it experiences bad return on its investments, an institution still generates value as soon as it doesn't default. This is the case in our setting for all institutions that don't experience the large shock, who can then support the institution suffering the shock. From an ex-ante point a view, each institution can then take more risk being in a cross-shareholding network rather than being isolated, and preserve the defaulting probability.[24]

24. To obtain Proposition 6, assuming $\mathbf{P}^T\mathbf{1} = \mathbf{P}\mathbf{1}$ is key.Otherwise, the existence of a non-empty cross-shareholding network can lead to a decrease in the risk-taking level of some institutions, due to the presence of a negative resource effect (see Appendix D).

5. Impact of VAR-RM on the expected shortfall

In this section, we examine how VaR-RM impacts the expected shortfall of a financial institution, a measure of the expected loss conditional on default. This measure therefore reflects the cost-of-default for the debt-holders, i.e. deposit insurance schemes in the case of banks.

Let $\tilde{\nu}_i(\tilde{\boldsymbol{\mu}}, \mathbf{z}) = x_i + \tilde{\mu}_i z_i - \rho d_i + \sum_{j \neq i} a_{ij} \tilde{v}_j(\tilde{\boldsymbol{\mu}}, \mathbf{z})$ represent the difference between asset and liability sides of institution i (see the balance sheet). Note that, in contrast to $\tilde{v}_i$, $\tilde{\nu}_i$ can take negative values. Under risk-taking vector $\mathbf{z}$, the expected shortfall of institution i is given by

$$ES_i(\mathbf{z}) = \mathbb{E}(\tilde{\nu}_i(\tilde{\boldsymbol{\mu}}, \mathbf{z}) | \tilde{\nu}_i(\tilde{\boldsymbol{\mu}}, \mathbf{z}) \leq 0)$$

This is the expected not-honored debt due to a default caused by the shock hitting the institution. This expression depends on the structure of the cross-shareholding network and on the nature of the shock hitting the financial system. To evaluate expected shortfalls when financial institutions comply to VaR-RM risk-taking $\mathbf{z}^*$, we need to specify the probability distribution functions of losses $\tilde{s}$. We consider in the following exponential and power law distributions. Recalling the equilibrium risk-taking $\mathbf{z}^*$ given by equation (9) and that $\ell_\beta = r - t_{1-\frac{n\beta}{q_0}}$, we obtain:

Proposition 7. *Consider Assumptions 1 to 4. Under Pareto distribution* $h_a(s) = \frac{a s_0^a}{s^{a+1}}$ *over the interval* $[s_0, +\infty)$*, for* $a > 1$*, the expected shortfall of institution* i *under VaR-RM is given by*

$$ES_i(\mathbf{z}^*) = \left(\frac{r - \ell_\beta}{a - 1}\right) m_{ii} z_i^*$$

Under exponential distribution $h_\lambda(s) = \lambda e^{-\lambda(s - s_0)}$ *over the interval* $[s_0, +\infty)$*, for* $\lambda > 0$*, the expected shortfall of institution* i *under VaR-RM is given by*

$$ES_i(\mathbf{z}^*) = \frac{m_{ii} z_i^*}{\lambda}$$

By Proposition 7, the cross-shareholding network has a sensible impact on institutions' expected shortfall. On the one hand, the network alleviates the shortfall by transmitting to the shocked firm the positive values of others; on the other hand through feedback loops it magnifies the losses in case of default. The last effect is increasing in the risk-taking level of the shocked institution. The impact of the cross-shareholding network on the expected shortfall sharply depends on the nature of the shock. Still, more cross-shareholding, i.e. augmenting matrix $\mathbf{A}$, has an unambiguous effect for the two considered distributions:

Corollary 2. *For both Pareto and exponential distributions of the shocks, network integration increases the expected shortfall of all financial institutions when the shock is localized ($\underline{r} = r$) and increases the average expected shortfall of financial institutions when the shock induces a stress in the financial system ($\underline{r} < 1$).*

This result is immediate using results from Proposition 2 and 4 as network integration increases matrix $\mathbf{M}$.

Table 6 illustrates the role of networks by comparing the average expected shortfall for various network structures. As for Table 5, we fix parameter values for probability distributions to share the same mean. Table 6 shows that both

	Star	Wheel	two-Core	Complete
Power law	7.3681	7.3699	7.3765	7.3861
Exponential	9.0944	9.0966	9.1047	9.1164

Figure 6. Average Expected Shortfall of the financial system for different networks and different probability distributions of the shock with same mean; $n = 6, \rho = 1.1, d = 1000, e = 1000, \underline{r} = 0.7, p = 1, q_0 = 0.1, \beta = 0.1, s_0 = 1, a = 10, \lambda = \frac{a-1}{s_0} = 9$.

network and nature of the shock have a strong impact on the average expected shortfall. It confirms that average expected shortfall increases with network integration for both shock distributions, as denser network are more likely to amplify the negative impact of the shock. Although denser networks lead to higher levels of investment (that is risk-taking) for the same probability of default (Proposition 2), they also entail larger expected shortfall, that is higher cost of default for depositors (and debt-holders).

This section highlights how risk-taking levels under VaR-RM, which requires the institution's default probability to remain sufficiently low, impacts debt-holder exposure to default. The use of VaR-RM, leading to linear interactions, provides a clean analytical treatment of interdependent risks. This analysis could serve as a foundation for studying risk management under an expected shortfall constraint. Indeed, choosing a level of risk-taking in response to other institutions' behavior is equivalent to selecting a default probability. This probability directly influences the expected shortfall, as it is given by the product of the expected loss and the default probability. In other words, selecting an optimal level of risk-taking to limit expected shortfall requires taking into account not only the associated default probability but also the strategic interdependencies among risk-taking behaviors of financial institutions that arise from this probability. Our results suggest that the strategic interactions among risk-taking behaviors we identified persists under such a risk management approach. However, conducting a fully analytical study in this setting is challenging. The relationship between an institution's default probability and its expected loss strongly depends on the specification of the probability distribution of shocks and is generally nonlinear for standard distributions.

6. Capital injection

Financial regulatory authorities face a trade-off between boosting investment and keeping default risks at acceptable levels. In this setting, they often prefer regulating the liability side of institutions' balance sheets (by setting capital requirements) rather than directly imposing constraints on institutions' investments in risky assets. To capture such policy, we allow for heterogeneous external equities $\mathbf{e} = (e_i)_{i\in\mathcal{I}}$ and heterogeneous debt $\mathbf{d} = (d_i)_{i\in\mathcal{I}}$. Then, in the same vein as equation (9), risk-taking under VaR-RM is written as

$$\mathbf{z}^* = \frac{1}{r-1}\Big[(1+\varepsilon_\beta)(\mathbf{I} - \varepsilon_\beta \mathbf{C})^{-1} - \mathbf{I}\Big](\mathbf{e} - (\rho - 1)\mathbf{d})$$

We can rewrite the above equation as a relationship between one institution's initial risky asset (at $t = 0$, through z_i) and its liability (through e_i):

$$\mathbf{e}(\mathbf{z}) = (1 - \ell_\beta)(\mathbf{I} - \mathbf{A})\mathbf{M}_\varepsilon \mathbf{z} + (\rho - 1)\mathbf{d} \tag{11}$$

where matrix $\mathbf{M}_\varepsilon$ has diagonal entry (i, i) equal to m_{ii} and off-diagonal entry (i, j) equal to $-\varepsilon_\beta m_{ij}$. Then, for any value of $\mathbf{z}$, $\mathbf{e}(\mathbf{z})$ represents the vector of capital requirements that keep default probabilities below a prescribed level (i.e., at value β). Regulatory capital requirement can then be understood as specific VaR-RM constraints. Equation (11) specifies the minimum external equity e_i an institution needs so as to be allowed to invest z_i in its risky asset. Here, the capital requirement for a given institution depends on its leverage, the overall risk profile, the cross-shareholding network, the value-at-risk ℓ_β, and the asset returns of other institutions when it receives a shock (r). Importantly, two firms with the same level of risk and leverage will not be required to hold the same level of capital if they have different network positions.

States or regulators can then wish to boost aggregate risky investments, that is aggregate expected return, while maintaining the same probability of default, by injecting capital into to some financial institutions. Network effects raise the question of the institution to target (we often refer to key-player analysis). Concretely, suppose that, while keeping default risk at the prescribed level corresponding to default probability β, the regulator chooses only one institution in which to inject equity, with the objective of maximizing aggregate investment in risky assets. The next proposition defines the institution that should be targeted. Defining matrix $\mathbf{W}$ such that $w_{ii} = m_{ii}$ and $w_{ij} = -\varepsilon_\beta m_{ij}$ for all i, j, and vector $\mathbf{w}^S = \mathbf{W}^{-1}\mathbf{1} = (w_i^S)_{i\in\mathcal{I}}$, the impact of adding one unit of external equity to institution i on the total investment in risky assets is given by

$$-\frac{1}{\varepsilon_\beta}+\left(\frac{1+\varepsilon_\beta}{\varepsilon_\beta}\right)m_{ii}w_i^S$$

We thus obtain:

Proposition 8. *Consider Assumptions 1 to 4. The institution to target is the one with the highest index $m_{ii}w_i^S$.*

Proposition 8 is useful to determine the optimal institution to target on the basis of network properties and the relative magnitude of the negative shock (through parameter ε_β) only. To illustrate, consider again the network depicted in Figure 2, and take the same parameters. Then the optimal target, that maximizes the index given in Proposition 8, is institution 1. Note that this index is not necessarily aligned with risk-taking. For instance, institution 3's index is higher than that of institution 4, whereas the ranking of respective risk-taking levels is reversed.

7. Conclusion

This study reveals that the structure of cross-shareholding networks significant impacts the risk-taking behaviors of financial institutions governed by Value-at-Risk Management (VaR-RM). Specifically, the nature of interactions among risk-taking decisions depends on the level of stress the financial system experiences following a shock. In the absence of stress, risk-taking are strategic complements; otherwise, they become strategic substitutes. We also find that, irrespective of whether an extreme event induces stress on the financial system, shareholding linkages increase risk-taking with respect to the no-network case. Moreover, denser networks always lead to more investment in risky assets on average. This positive effect of network integration is counterbalanced by a negative effect on expected-shortfall, meaning increased exposure to default for depositors when institutions use VaR-RM.

Our conclusions are based on the assumption of no contagion effects. However, in scenarios where the entire financial system faces systemic risk, financial institutions relying on VaR-RM may not fully account for their systemic risk exposure. Understanding how VaR-RM influences the vulnerability of the financial system to systemic crises and default contagion remains a complex issue for future research.

Moreover, our analysis shows the influence of the cross-shareholding network on the trade-off between risk-taking (that is, expected return) and default probability through VaR-RM approach. This trade-off is driven by the level of Value-at-Risk chosen. Our study thus opens avenue regarding the effect of network on optimal default probability, from an institution or a social point of view. Such an analysis would require additional assumptions on the costs of default and their distribution.

References

Acemoglu, D., Ozdaglar, A., and Tahbaz-Salehi, A. (2015). Systemic Risk and Stability in Financial Networks. *American Economic Review*, 105(2):564–608.

Allen, F. and Gale, D. (2000). Financial Contagion. *Journal of Political Economy*, 108(1):1–33.

Basak, A. and Shapiro, A. (2001). Value-at-risk-based risk management: Optimal policies and asset prices. *Review of Financial Studies*, 14(2):371–405.

BCBS (2019). The market risk framework: In brief. Basel committee on banking supervision.

Belhaj, M., Bramoullé, Y., and Deroïan, F. (2014). Network games under strategic complementarities. *Games and Economic Behavior*, 88(C):310–319.

Bodnar, G., Hayt, G., and Marston, R. (1998). Wharton survey of financial risk management by us non-financial firms. *Financial Management*, 27(4).

Bramoullé, Y., Kranton, R., and d'Amours, M. (2014). Strategic interaction and networks. *American Economic Review*, 104(3):898–930.

Brioschi, F., Buzzacchi, L., and Colombo, M. (1989). Risk capital financing and the separation of ownership and control in business groups. *Journal of Banking and Finance*, 13:747–772.

Cabrales, A., Gottardi, P., and Vega-Redondo, F. (2017). Risk Sharing and Contagion in Networks. *Review of Financial Studies*, 30(9):3086–3127.

Craig, B. and von Peter, G. (2014). Interbank tiering and money center banks. *Journal of Financial Intermediation*, 23(3):322–347.

Decamps, J.-P., Rochet, J.-C., and Roger, B. (2004). The three pillars of Basel II: optimizing the mix. *Journal of Financial Intermediation*, 13(2):132–155.

Demange, G. (2018). Contagion in financial networks: A threat index. *Management Science*, 64(2):955–970.

Dowd, K. (1998). *Beyond Value at Risk: the New Science of Risk Management*.

Eisenberg, L. and Noe, T. (2001). Systemic risk in financial systems. *Management Science*, 47(2):236–249.

Elliott, M., Golub, B., and Jackson, M. O. (2014). Financial Networks and Contagion. *American Economic Review*, 104(10):3115–3153.

Fedenia, M., Hodder, J., and Triantis, A. (1994). Cross-holdings: Estimation issues, biases, and distortions. *Review of Financial Studies*, 7(C):61–96.

Freixas, X. and Rochet, J.-C. (2013). Taming Systemically Important Financial Institutions. *Journal of Money, Credit and Banking*, 45:37–58.

Gai, P. and Kapadia, S. (2019). Networks and systemic risk in the financial system. *Oxford Review of Economic Policy*, 35(4):586–613.

Galeotti, A. and Ghiglino, C. (2021). Cross-ownership and portfolio choice. Technical Report 105194.

Glasserman, P. and Young, H. P. (2016). Contagion in Financial Networks. *Journal of Economic Literature*, 54(3):779–831.

Hellmann, T. F., Murdock, K. C., and Stiglitz, J. E. (2000). Liberalization, Moral Hazard in Banking, and Prudential Regulation: Are Capital Requirements Enough? *American Economic Review*, 90(1):147–165.

Jackson, M. O. and Pernoud, A. (2019). Distorted Investment Incentives, Regulation, and Equilibrium Multiplicity in a Model of Financial Networks. SSRN 3311839.

Johnson, C. R. and Smith, R. L. (2007). Positive, path product, and inverse m-matrices. *Linear Algebra and its Applications*, 421(2):328 – 337.

Jorion, P. (1998). Risk management lessons from long-term capital management.

Leduc, M. V. and Thurner, S. (2017). Incentivizing resilience in financial networks. *Journal of Economic Dynamics and Control*, 82(C):44–66.

Levine, R. (2005). Finance and Growth: Theory and Evidence. *Handbook of Economic Growth*, ed. by Philippe Aghion and Steven Durlauf, North Holland (Amsterdam: 2005).

Markowitz, H. (1952). Portfolio selection. *Journal of Finance*, 7(1):77–91.

Petra Valickova, T. H. and Horvath, R. (2015). Financial Development and Economic Growth: A Meta-Analysis. *Journal of Economic Surveys*, 29(4):506–526.

Pollak, M. and Guan, Y. (2017). Partially Overlapping Ownership and Contagion in Financial Networks. *Complexity*, 2017:1–16.

Rockafellar, T. and Uryasev, S. (2000). Optimization of conditional value-at-risk. *The Journal of Risk*, 2(3).

Rockafellar, T. and Uryasev, S. (2002). Conditional value-at-risk for general loss distributions. *The Journal of Banking and Finance*, 26(7):1443–1471.

Rotundo, G. and D'Arcangelis, A. (2014). Network of companies: an analysis of market concentration in the Italian stock market. *Quality & Quantity:*

International Journal of Methodology, 48(4):1893–1910.

Roy, A. (1952). Safety first and the holding of assets. *Econometrica*, 20:431–449.

Saunders, A. (1999). *Financial Institutions Management: A Modern Perspective*.

Shu, C. (2024). Endogenous Risk-Exposure and Systemic Instability. *Management Science*.

Appendix A: Proofs

The following lemma – reminiscent of Eisenberg and Noe (2001) – establishes the uniqueness of values satisfying the system of equations (2):

LEMMA A.A.1. *For any financial network* $(\mathbf{d}, \mathbf{e}, \mathbf{P})$*, any investment profile* $\mathbf{z} \in [\mathbf{0}, \mathbf{e} + \mathbf{d}]$*, and any realization of risks* $(\mu_i)_{i \in \mathcal{I}}$*, there is a single set of values* $\mathbf{v}$ *solving system (2) for all* i *(with possible defaults).*

Proof of Lemma A.A.1.

We define $\eta_i = e_i - (\rho - 1)d_i$, $h_i = (\mu_i - 1)z_i + \eta_i$ and $\mathbf{h} = (h_i)_{i \in \mathcal{I}}$. Equation (2) extended to heterogeneous e_i, d_i then simply writes:

$$v_i = \max(h_i + \sum_{j \neq i} a_{ij} v_j, 0) \tag{A.1}$$

that is, in the absence of default (if $v_i \geq 0$ for all i):

$$\mathbf{v} = \mathbf{M}\mathbf{h} \tag{A.2}$$

where $\mathbf{M} = (\mathbf{I} - \mathbf{A})^{-1}$. The largest eigenvalue of any sharing matrix $\mathbf{A}$ is lower than unity (as the sum of every column is lower than 1). Therefore, $(\mathbf{I} - \mathbf{A})^{-1} = \sum_{q=0}^{\infty} \mathbf{A}^q$.

Consider the system:

$$v_i = \max\left(0, h_i + \sum_{j\in\mathcal{I}} a_{ij} v_j\right) \quad \forall i \in \mathcal{I}$$

In v_is, this corresponds to a game of strategic complementarities with lower and upper bounds (for given μ_is), i.e. a supermodular game. Therefore, it possesses a minimum and a maximum equilibrium.

Now, consider an equilibrium with $\mathcal{S}$ non defaulting institutions, i.e. with $\mathbf{v}_{\mathcal{S}} = (v_1, \cdots, v_s) > 0$ and let $\bar{a}_i = 1 - \sum_{k\in\mathcal{S}} a_{ki}$. Then,

$$\sum_{i\in\mathcal{S}} \bar{a}_i v_i = \sum_{i\in\mathcal{S}} \left(1 - \sum_{k\in\mathcal{S}} a_{ki}\right) v_i$$

and given that $\sum_{i\in\mathcal{S}} v_i = \sum_{i\in\mathcal{S}} h_i + \sum_{i\in\mathcal{S}}\sum_{k\in\mathcal{S}} a_{ik} v_k$:

$$\sum_{i\in\mathcal{S}} \bar{a}_i v_i = \sum_{i\in\mathcal{S}} h_i$$

Last, suppose that the minimum equilibrium, say $\mathcal{S}$, is distinct from the maximum equilibrium, say $\mathcal{S}'$. Then $\mathbf{v}_{\mathcal{S}} < \mathbf{v}_{\mathcal{S}'}$ (we use here vectorial inequality) and

$$\sum_{i\in\mathcal{S}} h_i = \sum_{i\in\mathcal{S}} \bar{a}_i v_i < \sum_{i\in\mathcal{S}} \bar{a}_i v_i' < \sum_{i\in\mathcal{S}'} \bar{a}_i v_i' = \sum_{i\in\mathcal{S}'} h_i$$

However, by construction, for all institutions $i \in \mathcal{S}' \setminus \mathcal{S}$: $h_i < 0$. Indeed, by (A): $h_i > 0 \Rightarrow v_i > 0$ and all institutions with $h_i > 0$ always belong to the surviving set. Then, $\sum_{i\in\mathcal{S}'} h_i < \sum_{i\in\mathcal{S}} h_i$, which is in contradiction with (A). The equilibrium values are then unique.

□

The proof of Lemma A.A.1 rests on the complementarities between institutions' values, which, under multiplicity, would imply a minimum and a maximum configuration solving the system. Now, the total equity invested in the financial system is identical in both configurations, while there would be greater debt repayment in the maximum configuration, due to a larger number of survivors. This would leave less wealth to distribute in the maximum configuration than in the minimum configuration, despite the higher values in the maximum configuration. Hence, the two configurations coincide, implying uniqueness.[25]

Proof of Observation 1. Denote $\overline{p} = \sum_k p_{ki}$. We can write $\mathbf{A} = \mathbf{PW}$, where $\mathbf{W}$ is a diagonal matrix with diagonal entry $\mathbf{W}_{ii} = \frac{1}{P_i+e}$, with $P_i = \sum_k p_{ki}$. We have

$$\mathbf{b} = \mathbf{1} + (\mathbf{PW})\mathbf{1} + (\mathbf{PW})^2\mathbf{1} + \cdots$$

That is,

$$\mathbf{b} = \mathbf{1} + \mathbf{P}\Big(\mathbf{I} + \mathbf{WP} + (\mathbf{WP})^2 + \cdots\Big)\mathbf{W1}$$

Now, $\mathbf{WP} = \tilde{\mathbf{A}} = (\frac{p_{ij}}{P_i+e})$. Then,

$$\mathbf{b} = \mathbf{1} + \mathbf{P}(\mathbf{I} - \tilde{\mathbf{A}})^{-1}\mathbf{W1} \tag{A.3}$$

Note that, because $\sum_k p_{ki} = \sum_k p_{ki}$ $(= p_i)$,

$$\frac{1}{e}(\mathbf{I} - \tilde{\mathbf{A}})\mathbf{1} = \mathbf{W1} \tag{A.4}$$

Plugging (A.4) into (A.3), we get

$$\mathbf{b} = \mathbf{1} + \frac{1}{e}\mathbf{P1}$$

25. Complementarity in values also leads to a simple algorithm that pins down the equilibrium set of surviving institutions. Start with an initial set containing all institutions with positive constant h_i, and compute their values in this initial setting. Then extend the set by systematically testing neighbors as newcomers, and check whether each newcomer has a positive value. If so, include it in the set of survivors. This is an efficient algorithm: A newcomer to the current set of survivors never forces other survivors out of the set, which thus only expands during the process.

□

Proof of Theorem 1. The matrix form of the system of equations (8) is given by:

$$(\mathbf{I} - \varepsilon_\beta \mathbf{C})\mathbf{z} = \frac{e - (\rho - 1)d}{1 - \ell_\beta}(\mathbf{I} + \mathbf{C})\mathbf{1}$$

i.e.,

$$(\mathbf{I} - \varepsilon_\beta \mathbf{C})\mathbf{z} = \frac{e - (\rho - 1)d}{1 - \ell_\beta}\left[-\frac{1}{\varepsilon_\beta}(\mathbf{I} - \varepsilon_\beta \mathbf{C})\mathbf{1} + \frac{1 + \varepsilon_\beta}{\varepsilon_\beta}\mathbf{1}\right]$$

i.e., noting that $\frac{1}{(1-\ell_\beta)\varepsilon_\beta} = \frac{1}{r-1}$,

$$\mathbf{z} = \frac{e - (\rho - 1)d}{r - 1}\left[(1 + \varepsilon_\beta)(\mathbf{I} - \varepsilon_\beta \mathbf{C})^{-1}\mathbf{1} - \mathbf{1}\right]$$

Uniqueness is guaranteed by $e - (\rho - 1)d > 0$ and $\varepsilon_\beta > 0$ (see Belhaj et al., 2014), a direct implication from Assumption 3. Assumption 4 guarantees interiority.

□

Proof of Corollary 1. In our setting $\varepsilon_\beta \to 0$ when $|\ell_\beta|$ is large, that is when β is low, which corresponds to situations with tight risk-management. The Corollary stems from observing that

$$\lim_{\varepsilon_\beta \to 0} \mathbf{z}^* = \frac{e - (\rho - 1)d}{1 - \ell_\beta}(\mathbf{I} + \mathbf{C})\mathbf{1} \tag{A.5}$$

and by remarking that entry i of vector $(\mathbf{I} + \mathbf{C})\mathbf{1}$ is equal to $\frac{b_i}{m_{ii}}$. Recalling that $b_i = \frac{e+p_i}{e}$, the result follows.

□

Proof of Proposition 1. We proceed in two steps.

Step 1. Let us first show that the ratio b_i/m_{ii} is higher for the center of the star.

Consider a star network with n agents. We denote by 1 the center of the star and by 2 the representative periphery. As $\mathbf{P}^T\mathbf{1} = \mathbf{P}\mathbf{1} = (p_i)_{i\in\mathcal{I}}$, we have $b_1 = \frac{e+p_1}{e}$, $b_i = \frac{e+p_{1i}}{e}$, and for all $i > 1$, we have $p_{1i} = p_{i1}$, $a_{1i} = \frac{p_{1i}}{p_{1i}+e}$, $a_{i1} = \frac{p_{1i}}{p_1+e}$.

Also, $m_{11} = \frac{1}{det(\mathbf{I}-\mathbf{A})}$, and $m_{22} = \frac{1-\sum_{i>2} a_{1i}a_{i1}}{det(\mathbf{I}-\mathbf{A})}$.

Then $\frac{b_1}{m_{11}} > \frac{b_2}{m_{22}}$ if and only if

$$e + p_1 > \frac{e + p_{12}}{1 - \sum\limits_{i>2} a_{1i}a_{i1}}$$

Or,

$$(e + p_1)\left(1 - \sum_{i>2} \frac{p_{1i}^2}{(p_{1i} + e)(p_1 + e)}\right) > (e + p_{12}) \tag{A.6}$$

Inequality (A.6) is implied by

$$(e + p_1)\left(1 - \sum_{i>2} \frac{p_{1i}}{p_1 + e}\right) \geq (e + p_{12}) \tag{A.7}$$

That is, given that $p_1 - \sum\limits_{i>2} p_{1i} = p_{12}$,

$$(e + p_{12}) \geq (e + p_{12}) \tag{A.8}$$

which is true. This proves inequality (A.8), thus inequality (A.6).

Step 2. We now prove by induction that $z^*_{RS,1} > z^*_{RS,2}$, i.e. that the risk-taking level is higher for the center of the star than for any of the periphery.

To do so, we simply need to show that $\forall q$ $\left(\mathbf{C}^q\mathbf{1}\right)_1 > \left(\mathbf{C}^q\mathbf{1}\right)_2$. Now, by step 1, we know that $(\mathbf{C}\mathbf{1})_1 > (\mathbf{C}\mathbf{1})_2$. For convenience, let us $\psi_1 = (\mathbf{C}\mathbf{1})_1$, $\psi_2 = (\mathbf{C}\mathbf{1})_2$, and more generally, $\psi_1^{(q)} = (\mathbf{C}^q\mathbf{1})_1$, $\psi_2^{(q)} = (\mathbf{C}^q\mathbf{1})_2$ for all $q \geq 1$.

Let property $\mathcal{P}(q) : \varphi_c^{(q)} > \varphi_p^{(q)}$. Assume $\mathcal{P}(1), \cdots, \mathcal{P}(q-1)$. We will prove $\mathcal{P}(q)$. First note that

$$\psi_1^{(q)} = \psi_1 \psi_1^{(q-1)}$$

and

$$\psi_2^{(q)} = c_{21}\psi_1^{(q-1)} + (\psi_2 - c_{pc})\psi_2^{(q-1)}$$

The inequality $\psi_1^{(q)} > \psi_2^{(q)}$ then means

$$(\psi_1 - \psi_2)\, \psi_2^{(q-1)} > c_{21} \left(\psi_1^{(q-1)} - \psi_2^{(q-1)} \right) \tag{A.9}$$

Now, by $\mathcal{P}(q-1)$, we have

$$\psi_1 \psi_2^{(q-2)} > c_{21} \psi_1^{(q-2)} + (\psi_2 - c_{21}) \psi_2^{(q-2)}$$

and inequality (A.9) also writes

$$(\psi_1 - \psi_2)\psi_2^{(q-1)} > c_{21} \left((\psi_1 - \psi_2)\varphi_2^{(q-2)} - c_{21} \left(\psi_1^{(q-2)} - \psi_2^{(q-2)} \right) \right)$$

that is

$$(\psi_1 - \psi_2) \left(c_{21} \left(\psi_1^{(q-2)} - \psi_2^{(q-2)} \right) + (\psi_2 - c_{21})\psi_2^{(q-2)} \right) > -c_{21}^2 \left(\psi_1^{(q-2)} - \psi_2^{(q-2)} \right)$$

which holds whenever $\psi_2 - c_{21} > 0$. Now $\psi_2 > c_{21}$ corresponds to

$$\frac{\sum_{j \neq 2} m_{2j}}{m_{22}} > \frac{m_{21}}{m_{22}}$$

which always holds as $m_{ij} \geq 0 \; \forall i, j$. Therefore $\mathcal{P}(q)$ holds, whenever $\mathcal{P}(q-1)$ holds. As $\mathcal{P}(1)$ holds by Step 1, we have that regulated risk-taking is always higher for the center of the star than for the periphery.

□

Proof of Proposition 2. The following lemma shows an increasing relationship between matrix **A** and matrix **C**:

LEMMA A.A.2. *If* $\mathbf{A}' \leq \mathbf{A}$*, then* $\mathbf{C}' \leq \mathbf{C}$.

Proof of Lemma A.A.2. The proof relies on the Sherman-Morrison formula, that states: Suppose $\mathbf{Q}$ is an invertible n-square matrix with real entries and $\mathbf{r}, \mathbf{s} \in \mathbb{R}^n$ are column vectors. Then $\mathbf{Q} + \mathbf{r}\mathbf{s}^T$ is invertible if and only if

$1+\mathbf{s}^T\mathbf{Q}^{-1}\mathbf{r} \neq 0$. If $\mathbf{Q}+\mathbf{r}\mathbf{s}^T$ is invertible, its inverse is given by

$$(\mathbf{Q}+\mathbf{r}\mathbf{s}^T)^{-1} = \mathbf{s}^{-1} - \frac{\mathbf{Q}^{-1}\mathbf{r}\mathbf{s}^T\mathbf{Q}^{-1}}{1+\mathbf{s}^T\mathbf{Q}^{-1}\mathbf{r}} \tag{A.10}$$

We apply this formula with $\mathbf{Q} = \mathbf{I}-\mathbf{A}$ and $\mathbf{r}\mathbf{s}^T = -\mathbf{\Omega}$, where $\mathbf{\Omega} = [\omega_{ij}]$ is such that $\omega_{ij} = \omega$ if $(i,j) = (r,s)$, $\delta_{ij} = 0$ otherwise. Then matrix $\mathbf{\Omega}$ has a single non-zero entry, corresponding to a positive impulsion at the entry (r,s). It is easily shown that $\mathbf{\Omega} = -\mathbf{r}\mathbf{s}^T$ for $\mathbf{r} = (0,\cdots,0,\omega,0,\cdots,0)^T$ with ω at entry r, and $\mathbf{s}^T = (0,\cdots,0,1,0,\cdots,0)^T$ with 1 at entry s.

Applying the formula, noting $(\mathbf{I}-\mathbf{A})^{-1} = \mathbf{M}$ and $\mathbf{s}^T\mathbf{M}\mathbf{r} = -m_{rs}\omega$, we get

$$(\mathbf{I}-\mathbf{A}-\mathbf{\Omega})^{-1} = \mathbf{M} + \frac{\mathbf{M}\mathbf{\Omega}\mathbf{M}}{1-m_{rs}\omega}$$

Now the entry (i,j) of matrix $\mathbf{M}\mathbf{\Omega}\mathbf{M}$ is given by $[\mathbf{M}\mathbf{\Omega}\mathbf{M}]_{ij} = m_{ir}m_{sj}\omega$. Then,

$$[(\mathbf{I}-\mathbf{A}-\mathbf{\Omega})^{-1}]_{ij} = m_{ij} + \frac{m_{ir}m_{sj}\omega}{1-m_{sr}\omega}$$

We want to prove that the ratio $\frac{m_{ij}}{m_{ii}}$ increases for all i,j when $\mathbf{A}$ becomes $\mathbf{A}' = \mathbf{A}+\mathbf{\Omega}$. Note that $\frac{m_{ij}}{m_{ii}} \leq \frac{m_{ij}+a}{m_{ii}+b}$ if and only if $\frac{m_{ij}}{m_{ii}} \leq \frac{a}{b}$. Then it is sufficient to prove that $\frac{m_{ij}}{m_{ii}} \leq \frac{m_{ir}m_{sj}\omega}{m_{ir}m_{si}\omega}$, i.e.

$$\frac{m_{ij}}{m_{ii}} \leq \frac{m_{sj}}{m_{si}} \tag{A.11}$$

Now the path product property of any inverse M-matrix $\mathbf{Y}$ (see for instance Johnson and Smith, 2007, p. 329) writes

$$y_{ij}y_{jk} \leq y_{ik}y_{jj} \tag{A.12}$$

Equation (A.11) can be written:

$$m_{si}m_{ij} \leq m_{ii}m_{sj}$$

that is, permuting labels i and j:

$$m_{sj}m_{ji} \leq m_{jj}m_{si}$$

and, permuting labels i and q:

$$m_{ij}m_{js} \leq m_{jj}m_{is}$$

which corresponds to the path product property with i, j, s as shown by equation (A.12). $\mathbf{M}$ being an inverse M-matrix, we therefore have that $\mathbf{A}' > \mathbf{A}$ leads to $\mathbf{C}' > \mathbf{C}$, where $c_{ij} = m_{ij}/m_{ii}$ and $\mathbf{M} = (\mathbf{I} - \mathbf{A})^{-1}$.

□

By Lemma A.A.2, increasing the integration of the network of cross-shares induces an increase in the entries of matrix $\mathbf{C}$. The proof of Lemma A.A.2 relies on the path-product property of inverse M-matrices.[26] In particular, for any $p' \geq p$, we obtain $\mathbf{A}' \geq \mathbf{A}$ on a fixed network $\mathbf{G}$. That is, increasing the amount of investment from any existing investment increases the cross-share matrix.

We can now examine the impact of an increase in the cross-shareholding matrix $\mathbf{A}$ risk-taking under VaR-RM. At the interior equilibrium, the matrix $(\mathbf{I} - \varepsilon_\beta \mathbf{C})^{-1}$ is well-defined and nonnegative, so that $(\mathbf{I} - \varepsilon_\beta \mathbf{C})^{-1} = \sum_{k \geq 0} \varepsilon_\beta^k \mathbf{C}^k$. This implies that increased matrix $\mathbf{C}$ induces increased matrix $(\mathbf{I} - \varepsilon_\beta \mathbf{C})^{-1}(\mathbf{I} + \mathbf{C})$. Therefore, the risk-taking $\mathbf{z}^*$ under VaR-RM increases when cross-shareholding increases.

□

Proof of Proposition 4. Suppose $\underline{r} < 1$, that is $\varepsilon_\beta < 0$. The optimal vector of risk-taking solves

$$(\mathbf{I} + |\varepsilon_\beta|\mathbf{C})\mathbf{z} = \frac{e - (\rho - 1)d}{1 - \ell_\beta}(\mathbf{I} + \mathbf{C})\mathbf{1}$$

i.e.,

$$(\mathbf{I} + |\varepsilon_\beta|\mathbf{C})\mathbf{z} = \frac{e - (\rho - 1)d}{(1 - \ell_\beta)|\varepsilon_\beta|}\Big[(\mathbf{I} + |\varepsilon_\beta|\mathbf{C})\mathbf{1} - (1 - |\varepsilon_\beta|)\mathbf{1}\Big]$$

26. An M-matrix is a n-by-n matrix with non-positive off-diagonal entries and has an entry-wise non-negative inverse. In our case, $\mathbf{M} = (\mathbf{I} - \mathbf{A})^{-1}$ is then an inverse M-matrix.

i.e., inverting matrix $(\mathbf{I}+|\varepsilon_\beta|\mathbf{C})$,

$$\mathbf{z} = \frac{e-(\rho-1)d}{(1-\ell_\beta)|\varepsilon_\beta|}\Big[\mathbf{1}-(1-|\varepsilon_\beta|)(\mathbf{I}+|\varepsilon_\beta|\mathbf{C})^{-1}\mathbf{1}\Big] \tag{A.13}$$

We are now in position to evaluate how increased integration affects risk-taking. By Lemma A.A.2, an increase in the shareholding matrix $\mathbf{A}$ entails an increase in matrix $\mathbf{C}$. By equation (A.13), risk-taking is a decreasing function of the solution of a linear system of substitute interaction, represented here by the term $(\mathbf{I}+|\varepsilon_\beta|\mathbf{C})^{-1}\mathbf{1}$. It is well-known that, in a classical system of linear interaction with strategic substitutes, increasing interaction decreases the average output. Thus, increased matrix $\mathbf{C}$ entails a decrease of the average of the vector $(\mathbf{I}+|\varepsilon_\beta|\mathbf{C})^{-1}\mathbf{1}$, which implies an increase in average risk-taking. □

Proof of Proposition 5. Consider (A.13), that is

$$\mathbf{z} = \frac{e-(\rho-1)d}{(1-\ell_\beta)}\Big[\frac{1}{|\varepsilon_\beta|}\mathbf{1}+\Big(1-\frac{1}{|\varepsilon_\beta|}\Big)(\mathbf{I}+|\varepsilon_\beta|\mathbf{C})^{-1}\mathbf{1}\Big]$$

Let

$$\mathbf{h}(|\varepsilon_\beta|) = \frac{1}{|\varepsilon_\beta|}\mathbf{1}+\Big(1-\frac{1}{|\varepsilon_\beta|}\Big)(\mathbf{I}+|\varepsilon_\beta|\mathbf{C})^{-1}\mathbf{1}$$

Observing that ℓ_β is increasing in β and $|\varepsilon|$ is decreasing in β (as $\varepsilon<0$), to prove that the average risk-taking is increasing in β it is sufficient to show that $h(|\varepsilon|) = \mathbf{1}^T\mathbf{h}(|\varepsilon|)$ is decreasing in $|\varepsilon|$.

Consider $|\varepsilon'|>|\varepsilon|$. Note that $\mathbf{h}(|\varepsilon|) = \frac{1}{|\varepsilon|}\mathbf{1}+\mathbf{x}(|\varepsilon|)$, where

$$(\mathbf{I}+|\varepsilon|\mathbf{C})\mathbf{x}(|\varepsilon|) = \Big(1-\frac{1}{|\varepsilon|}\Big)\mathbf{1}$$

Note also that

$$\big(\mathbf{I}+|\varepsilon|\mathbf{C}+(|\varepsilon'|-|\varepsilon|)\mathbf{C}\big)\mathbf{x}(|\varepsilon'|) = \Big(1-\frac{1}{|\varepsilon'|}\Big)\mathbf{1}$$

That is,

$$(\mathbf{I}+|\varepsilon|\mathbf{C})\mathbf{x}(|\varepsilon'|)=\Big(1-\frac{1}{|\varepsilon'|}\Big)\mathbf{1}-(|\varepsilon'|-|\varepsilon|)\mathbf{C}\mathbf{x}(|\varepsilon'|)$$

Then,

$$(\mathbf{I}+|\varepsilon|\mathbf{C})(\mathbf{x}(|\varepsilon'|)-\mathbf{x}(|\varepsilon|))=\frac{|\varepsilon'|-|\varepsilon|}{|\varepsilon||\varepsilon'|}\mathbf{1}-(|\varepsilon'|-|\varepsilon|)\mathbf{C}\mathbf{x}(|\varepsilon'|)$$

Let $x(|\varepsilon|)=\mathbf{1}^T\mathbf{x}(|\varepsilon|)$, and let $\mathbf{w}=(\mathbf{I}+|\varepsilon|\mathbf{C})^{-1}\mathbf{1}\in(0,1)$; hence $w=\mathbf{1}^T\mathbf{w}<n$. We then find

$$\mathbf{x}(|\varepsilon'|)-\mathbf{x}(|\varepsilon|)=\frac{|\varepsilon'|-|\varepsilon|}{|\varepsilon||\varepsilon'|}\mathbf{w}-(|\varepsilon'|-|\varepsilon|)(\mathbf{I}+|\varepsilon|\mathbf{C})^{-1}\mathbf{C}\mathbf{x}(|\varepsilon'|)$$

And summing over entries,

$$x(|\varepsilon'|)-x(|\varepsilon|)=\frac{|\varepsilon'|-|\varepsilon|}{|\varepsilon||\varepsilon'|}w-(|\varepsilon'|-|\varepsilon|)\mathbf{w}^T\mathbf{C}\mathbf{x}(|\varepsilon'|)$$

Hence,

$$h(|\varepsilon'|)-h(|\varepsilon|)=\frac{|\varepsilon'|-|\varepsilon|}{|\varepsilon||\varepsilon'|}(w-n)-(|\varepsilon'|-|\varepsilon|)\mathbf{w}^T\mathbf{C}\mathbf{x}(|\varepsilon'|)$$

We deduce that $h(|\varepsilon'|)-h(|\varepsilon|)<0$.

□

Proof of Proposition 6. Consider that institution i is hit by the shock. By equation (2), the value of a surviving institution i exerting risk-taking level z_i, and given that others' VaR-RM risk-taking level, is given by

$$v_i(z_i)=f_i(z_i)+g(z_i)$$

where $f_i(z_i)=(\ell_\beta-1)z_i+e-(\rho-1)d$, and where $g(z_i)=\sum_{j\neq i}a_{ij}v_j^+(z_j^*(z_i))\geq 0$.

When there is no network, VaR-RM entails $f_i(z_i^0)=0$; While, when there is a network, VaR-RM entails $f_i(z_i^*)=-g(z_i^*)\leq 0$. Since $\ell_\beta<1$, function f_i is decreasing, which implies that $z_i^0\leq z_i^*$.

□

Proof of Proposition 7. Under risk-taking vector $\mathbf{z}$, the expected shortfall of institution i is given by

$$ES_i(\mathbf{z}) = \mathbb{E}(\tilde{\nu}_i(\mathbf{z})|\tilde{\nu}_i(\mathbf{z}) < 0)$$

Conditional on i suffering the shock (the only possible state for which $\tilde{\nu}_i(\mathbf{z}) < 0$):

$$\tilde{\nu}_i(\mathbf{z}) = (r-1)\sum_{k\in\mathcal{I}} m_{ik}z_k + b_i\eta - m_{ii}z_i\tilde{s}$$

and denoting

$$\Gamma_i(\mathbf{z}) = (r-1)\sum_{k\in\mathcal{I}} m_{ik}z_k + b_i(e-(\rho-1)d)$$

we have $\nu_i = 0$ for a realization of the shock $s_i^0(\mathbf{z})$ such that

$$s_i^0(\mathbf{z}) = \frac{\Gamma_i(\mathbf{z})}{m_{ii}z_i}$$

The expected value of an institution i, conditional on defaulting, is thus given by

$$\mathbb{E}(\tilde{\nu}_i|\tilde{\nu}_i < 0) = \Gamma_i(\mathbf{z}) - m_{ii}z_i\mathbb{E}\left(\tilde{s}|\tilde{s} > \frac{\Gamma_i(\mathbf{z})}{m_{ii}z_i}\right)$$

Hence, the expected shortfall of institution i is written as

$$ES_i(\mathbf{z}) = -\Gamma_i(\mathbf{z}) + m_{ii}z_i\mathbb{E}\left(\tilde{s}|\tilde{s} > \frac{\Gamma_i(\mathbf{z})}{m_{ii}z_i}\right)$$

We explore now the impact of risk-taking behavior $\mathbf{z}^*$, issued from VaR-RM, on the expected shortfall, for both Pareto and exponential probability distributions.

- Assume that the shock $\tilde{s}$ has a Pareto density distribution over $[s_0, +\infty)$: $h_a(s) = \frac{as_0^a}{s^{a+1}}$ for $a > 1$. Then it is well-known that

$$\mathbb{E}(\tilde{s}|\tilde{s} > s_i^0(\mathbf{z})) = \frac{as_i^0(\mathbf{z})}{a-1}$$

This implies that

$$\mathbb{E}(\tilde{\nu}_i|\tilde{\nu}_i < 0) = \Gamma_i(\mathbf{z}) - m_{ii}z_i \cdot \frac{as_i^0(\mathbf{z})}{a-1}$$

which, given $m_{ii}z_i s_i^0 = \Gamma_i(\mathbf{z})$, is simplified as

$$\mathbb{E}(\tilde{\nu}_i|\tilde{\nu}_i < 0) = -\frac{\Gamma_i(\mathbf{z})}{a-1}$$

We therefore get, for the Pareto distribution of the shock of parameter a,

$$ES_i(\mathbf{z}^*) = \frac{1}{a-1}\Gamma_i(\mathbf{z}^*)$$

Recalling that $\ell_\beta = r - t_{1-\frac{n\beta}{q_0}}$ and that the FOC defining VaR-RM risk-taking gives $\Gamma_i(\mathbf{z}^*) = t_{1-\frac{n\beta}{q_0}} \cdot m_{ii}z_i^*$, we deduce

$$ES_i(\mathbf{z}^*) = \left(\frac{t_{1-\frac{n\beta}{q_0}}}{a-1}\right) m_{ii}z_i^*$$

• Assume that the shock $\tilde{s}$ has an exponential density distribution: $h_\lambda(s) = \lambda e^{-\lambda(s-s_0)}$ for $\lambda > 0$. Then

$$\mathbb{E}(\tilde{s}|\tilde{s} > s_i^0(\mathbf{z})) = s_i^0(\mathbf{z}) + \frac{1}{\lambda}$$

This implies that

$$\mathbb{E}(\tilde{\nu}_i|\tilde{\nu}_i < 0) = \Gamma_i(\mathbf{z}) - m_{ii}z_i \cdot \left(s_i^0(\mathbf{z}) + \frac{1}{\lambda}\right)$$

which, given $m_{ii}z_i s_i^0(\mathbf{z}) = \Gamma_i(\mathbf{z})$, is simplified as

$$\mathbb{E}(\tilde{\nu}_i|\tilde{\nu}_i < 0) = -\frac{m_{ii}z_i}{\lambda}$$

We thus obtain, for the exponential distribution of parameter λ,

$$ES_i(\mathbf{z}^*) = \frac{m_{ii}z_i^*}{\lambda}$$

□

Proof of Proposition 8. Defining $v_i = \frac{1}{1-\ell_\beta}\Big(m_{ii}\eta_i + \sum_{j\neq i} m_{ij}\eta_j\Big)$, the initial $\mathbf{z}^*$ solves

$$m_{ii}z_i^* - \varepsilon\sum_{j\neq i} m_{ij}z_j^* = v_i$$

Or, in matrix notation,

$$\mathbf{W}\mathbf{z}^* = \boldsymbol{v}$$

where $\mathbf{W}$ is a n-dimensional square matrix such that $w_{ii} = m_{ii}$ and $w_{ij} = -\varepsilon m_{ij}$; and $\boldsymbol{v} = (v_i)_{i\in\mathcal{I}}$.

Suppose now that one $1-\ell_\beta$ unit of cash in the external equity of institution 1 (for ease of exposition, all the following addresses institution i). Letting $\mathbf{m}_1 = (m_{11}, m_{21}, \cdots, m_{n1})^T$ be the first column of matrix $\mathbf{M}$, the optimal risk-taking $\mathbf{z}'^*$ then writes

$$\mathbf{W}\mathbf{z}'^* = \boldsymbol{v} + \mathbf{m}_1$$

and the change in total investment in the risky asset is

$$\mathbf{1}^T(\mathbf{z}' - \mathbf{z}) = \mathbf{1}^T\mathbf{W}^{-1}\mathbf{m}_1$$

Observing that

$$\mathbf{m}_1 = -\frac{1}{\varepsilon}\begin{pmatrix} m_{11} \\ -\varepsilon m_{21} \\ \dots \\ -\varepsilon m_{n1}\end{pmatrix} + \frac{1+\varepsilon}{\varepsilon}\begin{pmatrix} m_{11} \\ 0 \\ \dots \\ 0\end{pmatrix}$$

we obtain

$$\mathbf{W}^{-1}\mathbf{m}_1 = -\frac{1}{\varepsilon}\begin{pmatrix} 1 \\ 0 \\ \dots \\ 0\end{pmatrix} + \frac{1+\varepsilon}{\varepsilon}\mathbf{W}^{-1}\begin{pmatrix} m_{11} \\ 0 \\ \dots \\ 0\end{pmatrix}$$

Thus, defining $\mathbf{1}^T\mathbf{W}^{-1} = (w_1^S, w_2^S, \cdots, w_n^S)$, so that w_i^S is the sum of entries of column i in matrix $\mathbf{W}^{-1}$, we obtain that

$$\mathbf{1}^T(\mathbf{z}' - \mathbf{z}) = -\frac{1}{\varepsilon} + \left(\frac{1+\varepsilon}{\varepsilon}\right) m_{11} w_1^S$$

The highest effect on total investments in risky assets ($\mathbf{1}^T(\mathbf{z}' - \mathbf{z})$) is achieved by targeting for capital injection the institution with the highest index $m_{ii}w_i^S$.

□

Appendix B: Optimal risk-taking in the absence of VaR-RM

In this appendix, we explore the behavior of financial institutions in the absence of Value-at-Risk Management. Each institution is risk neutral and maximizes its expected equity value $\mathbb{E}(v_i)$.[27] In this setup, obviously firms take more risk than under VaR-RM. However, the presence of the extreme event may lead them not to put all resource in the risky asset. We give an upper bound on the probability of the bad event under which institutions still put all their resource in the risky asset.

In this setup, the institution's decision consists in allocating its resources between the risk free asset and its specific risky asset. Using equation (2), this comes to:

$$\max_{z_i \in [0, e+d]} \mathbb{E}\left(\max\left((\tilde{\mu}_i - 1)z_i + e - (\rho - 1)d + \sum_{j \neq i} a_{ij}\tilde{v}_j, 0 \right) \right) \tag{B.1}$$

In the above expression, the realizations of any $\tilde{v}_j$ is necessarily nonnegative through equation (2).

Even if the average return on the risky asset is larger than the one of the risk-free asset, through network effects, risk-neutral institutions may not want

27. Managers' and equity-holders' objectives are assumed to be aligned. We thus ignore agency issues inside the institution.

to put all their resource to the risky asset. This phenomenon arises when the probability that the shock hits the financial system, q_0, is high. Indeed, on top of the classical direct effect (the first term of (B.1), positive under Assumption 2), the level of risk-taking by one institution also impacts its value through self-loops in the risk-sharing network. This last effect can dominate when the probability of shock q_0 is large. As we want to focus here – consistently with current regulation – on extreme event that occurs with low probability, we assume that this q_0 is low enough (Assumption B.1), so that the first classical effect dominates.

Assumption B.1. $q_0 \leq \frac{1}{1-\frac{1}{n}\left(\frac{r-\overline{s}-1}{r-1}\right)}$.

Indeed:

Proposition B.B.1. *Under Assumptions 1, 2 and B.1, the expected value of a financial institution is increasing with its risk-taking level. Then unregulated institutions optimally allocate all their resources to the risky asset:* $z_i^{*u} = e + d$ $\forall i$.

By Proposition B.B.1, institutions invest their whole resource in the risky asset when the probability that the negative shock hits the financial system is sufficiently low.

Proof of Proposition B.B.1.

Lemma B.B.1. *For all* μ, $\mathbf{z}, \mathbf{z}' = (z_i', z_{-i})$ *such that* $z_i \leq z_i'$,

$$v_i(\mathbf{z}') - v_i(\mathbf{z}) \geq (\mu_i - 1)m_{ii}(z_i' - z_i)$$

Proof of Lemma B.B.1. Call $\mathcal{I}$ the set of surviving institutions under $(\mu, \mathbf{z})$, $\mathcal{I}'$ the set of surviving institutions under $(\mu, \mathbf{z}')$, and $\mathbf{M}$ and $\mathbf{M}'$ the respective invert matrices of the systems. We have

$$\begin{cases} v_i = \sum_{j\in\mathcal{I}} m_{ij}((\mu_j-1)z_j + e - (\rho-1)d) \\ v_i' = \sum_{j\in\mathcal{I}'} m_{ij}'((\mu_j-1)z_j' + e - (\rho-1)d) \end{cases}$$

Given the structure of risk of the model, there are three cases to consider. Either the change in institution i's risk-taking does not affect the set of surviving institutions (Case (i)), or it implies one more surviving institution (Case (ii)), or it implies one less surviving institution (Case (iii)).

Case (i) $\mathcal{I} = \mathcal{I}'$. Then $\mathbf{M}' = \mathbf{M}$, and:

$$v_i' - v_i = (\mu_i - 1)m_{ii}(z_i' - z_i)$$

Case (ii): $\mu_i > 1$ and $\mathcal{I}' = \mathcal{I} \cup \{k\}$. Hence, $\mathbf{M} \le \mathbf{M}'(\mathcal{I}')$. Consider the level $z_i^c \in (z_i, z_i')$, at which institution k becomes knife-edge, i.e. such that its value is equal to zero under both systems $\mathcal{I}$ and $\mathcal{I}'$. Such a value exists by continuity. Denote by v_i^c the value of institution i at (z_i^c, z_{-i}). Then,

$$v_i' - v_i = v_i' - v_i^c + v_i^c - v_i$$

Consider $v_i^c - v_i$. We are here in Case (i), and then

$$v_i^c - v_i = (\mu_i - 1)m_{ii}(z_i' - z_i)$$

Now consider $v_i' - v_i^c$. We are here in Case (i) again, but with matrix $\mathbf{M}'$; we deduce

$$v_i' - v_i^c = (\mu_i - 1)m_{ii}'(z_i' - z_i)$$

Therefore,

$$v_i' - v_i = (\mu_i - 1)\Big(m_{ii}'(z_i' - z_i^c) + m_{ii}(z_i^c - z_i)\Big)$$

And since $\mu_i > 1$ and $m_{ii} < m'_{ii}$, we obtain

$$v'_i - v_i \geq (\mu_i - 1)m_{ii}(z'_i - z_i)$$

Case (iii): $\mu_i < 1$ and $\mathcal{I} = \mathcal{I}' \cup \{k\}$. Hence, $\mathbf{M} \geq \mathbf{M}'(\mathcal{I})$; note that $\mu_i < 1$ cannot induce that higher risk-taking from i hurts institution k's health. Consider the level $z_i^c \in (z_i, z'_i)$, at which institution k becomes knife-edge, i.e. such that its value equal to zero under both systems $\mathcal{I}$ and $\mathcal{I}'$ (like Case (ii), such a value exists by continuity). Replicating the same argument as Case (ii), we find

$$v'_i - v_i = (\mu_i - 1)\Big(m'_{ii}(z'_i - z_i^c) + m_{ii}(z_i^c - z_i)\Big)$$

And since $\mu_i < 1$ and $m_{ii} > m'_{ii}$, we obtain

$$v'_i - v_i \geq (\mu_i - 1)m_{ii}(z'_i - z_i)$$

□

By Lemma B.B.1, following an increase in z_i (from z_i to z'_i), the gap in the expected value of institution i is bounded from below; i.e., denoting $\Delta z_i = z'_i - z_i$ and $\underline{m}_{ii} = \min_{j \neq i} m_{ii}^{-j}$:

$$\mathbb{E}(\tilde{v}'_i) - \mathbb{E}(\tilde{v}_i) \geq (1 - q_0)(r - 1)m_{ii}\Delta z_i + \frac{q_0}{n}(r - \overline{s} - 1)m_{ii}\Delta z_i + \frac{q_0(n-1)}{n}(r - 1)\underline{m}_{ii}\Delta z_i \quad \text{(B.2)}$$

In the RHS, the first term corresponds to the no-shock case, the second term corresponds to institution i being hit by the shock, and the third term corresponds to another institutions being hit. Importantly, this is adapted from Case (ii) in lemma B.B.1 and leads to bound the value from below with the term $\underline{m}_{ii}$, which is such that $\underline{m}_{ii} < m_{ii}$ (under return greater than unity, the self-loop of institution i allowing to give a lower bound is that of the smallest network).

Since $r > 1$, from inequality (B.2), a sufficient condition for $\mathbb{E}(\tilde{v}_i') - \mathbb{E}(\tilde{v}_i) \geq 0$ (after dropping the third negative term and the negative quantities associated with the unit return in both first and second term) is given by

$$(1 - q_0)(r - 1) + \frac{q_0}{n}(r - \overline{s} - 1) \geq 0$$

That is,

$$q_0 \leq \frac{1}{1 - \frac{1}{n}\left(\frac{r - \overline{s} - 1}{r - 1}\right)}$$

□

Appendix C: Multiple shocks

This appendix presents a possible modeling of risk management under multiple shocks hitting the financial system. The overall model generates complementarities in risk-taking levels, but multiple shocks bring equilibrium multiplicity. Even under equilibrium multiplicity, the comparative statics presented in the single-shock case, like Proposition 2, still generically hold at least locally.

In this extension, the catastrophic event affects $q + 1$ institutions at the same time, with $q \in \{0, 1, \cdots, n - 1\}$ ($q = 0$ in the benchmark model with a single shock),[28] but the shock hits the financial system at random with uniform probability across institutions. A prudent risk management imposes an upper bound on the default probability of each bank conditionally on being shocked and any other q banks shocked and defaulted.[29] This objective leads to the

28. For simplicity, this value is assumed to be common knowledge among institutions.

29. Alternatively, firms may want to bound the *unconditional* probability of default, rather than the probability of default conditional to the worst state of nature. In this case, institutions should take into account the default probabilities of other shocked institutions. This alternative scenario can hardly be explored analytically.

following managerial constraints:

$$\mathbb{P}(\tilde{v}_i < 0 | v_{k_1} = 0, \cdots, v_{k_q} = 0) \leq \beta \ \forall i, \forall \{k_1, \cdots, k_q\} \subset \mathcal{I} \setminus \{i\} \tag{C.1}$$

where all institutions in $\{k_1, \cdots, k_q\}$ are shocked and defaulted. Hence, institution i should survive with probability β to a negative shock hitting it with certainty plus any q other simultaneous shocks hitting other institutions. We define the set of critical institutions to institution i as the set of institutions such that the above equation is binding.

Conforming to the worst-case-scenario basis of VaR-RM, critical institutions to any institution i are those whose dropout hurts institution i's expected value the most.[30] The set of critical institutions of any institution i, as well as institution i's best-response risk-taking, are determined jointly. We define the finite set $\mathcal{S}_i$ of all subsets of q distinct institutions out of the set $\mathcal{I} \setminus \{i\}$; $\mathcal{S}_i = \emptyset$ in the single-shock case $q = 0$.

To evaluate how the dropout of a given group of shocked institutions $\mathcal{S} \in \mathcal{S}_i$ affects the value of institution i, we need to take into account that the dropout restricts the interactions system generating institution i's best-response risk-taking. To take into account that a defaulting institution does not transmit any value to others (particularly a negative value), we introduce the modified cross-shareholding matrix $\mathbf{A}^{\mathcal{S}}$, in which each share invested in a defaulting institution in the set $\mathcal{S}$ is put to zero; that is, for every institution $k \in \mathcal{I}$, for all $j \in \mathcal{S}$, $a^{\mathcal{S}}_{kj} = 0$. We denote by $\mathbf{C}^{\mathcal{S}}$ the analogous matrix to matrix $\mathbf{C}$ associated with cross-shareholding matrix $\mathbf{A}^{\mathcal{S}}$. When the shocked institutions are in the

30. In what follows, we will abuse the notation by assuming a single maximizor; under multiple maximizors, choosing any set among them is indifferent.

set $\mathcal{S}$, equation (C.1) becomes

$$q_1.\mathbb{P}\left((r-\tilde{s}-1)z_i+e-(\rho-1)d+\sum_{j\in\mathcal{N}\setminus\mathcal{S}}c_{ij}^{\mathcal{S}}\left[(r-1)z_j+e-(\rho-1)d\right]<0\right)\leq\beta \tag{C.2}$$

where $q_1 = q_0\binom{n}{q-1}$ (recall that q_0 represents the probability that the shocks hit the financial system).

Recalling that $t_{1-\frac{n\beta}{q_1}}$ is the $(1-\frac{n\beta}{q_1})$th quantile of the distribution of $\tilde{s}$ and $\ell_\beta = r - t_{1-\frac{n\beta}{q_1}}$, and taking into account that equation (C.2) is binding at the optimum, we can determine the risk-taking of institution i, $z_i^*(\mathbf{z}_{-i})$, which makes condition (C.1) binding:

$$z_i^*(\mathbf{z}_{-i})=\frac{1}{1-\ell_\beta}\left(e-(\rho-1)d+\min_{\mathcal{S}\in\mathcal{S}_i}\sum_{j\in\mathcal{I}\setminus\mathcal{S}}c_{ij}^{\mathcal{S}}\Big((r-1)z_j+e-(\rho-1)d\Big)\right) \tag{C.3}$$

By equation (C.3), optimal risk-taking decisions are still strategic complements (as in the case of a single shock). Like the case of a single shock, the institutions that survive in the network always provide support to institution i through cross-shareholding links. However, with multiple shocks, each institution has its own relevant network of complementarities, induced from the whole cross-shareholding network by dropping its set of critical institutions.

Even if strategic complementarities resist the introduction of shock multiplicity, system (C.3) is highly non-linear, and both cycles and equilibrium multiplicity may emerge, as illustrated by the six-institution example shown in Fig. C.1. Consider the fixed-participation case and the following parameters: $q = 1$, $\rho = 1.01$, $d = 1000$, $e = 100$, $l = 0.85$, $r = 1.02$, and $p = 5$. Consider a sequential best-response algorithm (SBRA) with discrete periods, where a single institution reacts at a time in any pre-definite order, starting from any initial risk-taking vector. A Nash equilibrium is a fixed point of such a SBRA. Then, numerical computations

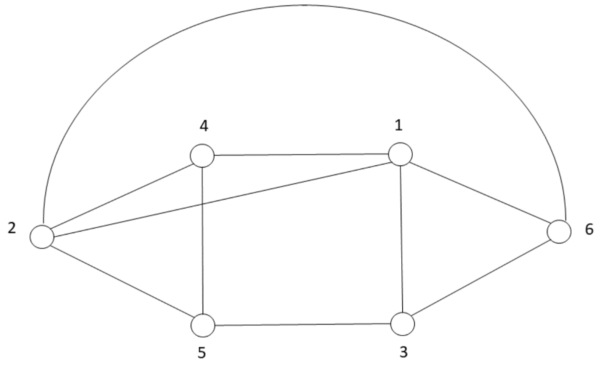

FIGURE C.1. Two shocks in the economy ($q = 1$). This six-institution network can generate multiple equilibria.

show that both $\mathbf{z}_1^* = (697.45, 694.19, 631.33, 663.97, 659.83, 662.67)$ and $\mathbf{z}_2^* = (697.51, 664.49, 664.36, 631.33, 662.67, 659.83)$ are fixed points of the SBRA. If equilibrium multiplicity asks for the question of equilibrium selection, there is no simple answer, because, as suggested in the above examples, equilibria might not be ranked.

Lastly, even under equilibrium multiplicity, the comparative statics presented in the single-shock case, like Proposition 2, still hold at least locally; that is, for any change of parameter sets that keeps unchanged the set of critical institutions for each institution, the system of interactions describing institutions' values, and thus risk-raking levels, is of the same qualitative nature as for the single-shock benchmark (i.e., complementarities), so our proofs extend straightforwardly.

Appendix D: Directed cross-shareholding network

In this section, we allow for cross-investments to be bilaterally asymmetric; i.e., $p_{ij} \neq p_{ji}$ is now possible. We examine the impact on risk-taking and on the comparative statics on integration. Under asymmetric cross-shareholding, contagion of a single default is possible, thus we assume that bilateral differences in cross-investment are sufficiently low as compared to leverage ratio so as to avoid contagion effects (see Assumption D.1).

With asymmetric bilateral relationships, we obtain the following accounting equation at $t = 0$:

$$x_i + z_i + \sum_{j \in \mathcal{I}} p_{ij} = d + e + \sum_{j \in \mathcal{I}} p_{ji} \tag{D.1}$$

meaning that the accounting equation at $t = 1$ becomes

$$v_i = \max\Big((\mu_i - 1)z_i + e - (\rho - 1)d + \underbrace{\sum_{j \in \mathcal{I}} (p_{ji} - p_{ij})}_{\text{Resource effect}} + \sum_{j \neq i} a_{ij} v_j, 0\Big) \tag{D.2}$$

The additional term is a resource effect, by which institutions with more investors from the financial system benefit from more resource to allocate between risk-free and risky asset.

We impose a generalized version of Assumption 1 as follows.

Assumption D.1. $e - (\rho - 1)d + \sum_{j \in \mathcal{I}} (p_{ji} - p_{ij}) > 0$ for all i

Under Assumption D.1, when an institution does not invest in risky assets, it remains solvent (i.e., $v_i > 0$ when $z_i = 0$). When the network of cross-shareholding is balanced ($\sum p_{ji} = \sum p_{ij}$), Assumption D.1 reduces to the condition $e > (\rho - 1)d$ for all i (i.e., Assumption 1), meaning that banks' equity suffices to finance the interest paid on debt. More generally, this assumption also depends on $\sum p_{ji} - \sum p_{ij}$, hereafter called the resource effect, which must be of sufficiently low magnitude.

Risk-taking under VaR-RM. The expression of interior regulated risk-taking is the same as that of Theorem 1, except for the resource effect (i.e., the term $(\mathbf{P}^T - \mathbf{P})\mathbf{1}$ hereafter):

$$\mathbf{z}^* = \frac{1}{1-\ell_\beta}(\mathbf{I} - \varepsilon\mathbf{C})^{-1}(\mathbf{I} + \mathbf{C})(e - (\rho - 1)d)\mathbf{1} + (\mathbf{P}^T - \mathbf{P})\mathbf{1}) \qquad \text{(D.3)}$$

We now analyze the characteristics of the centrality measure $\mathbf{z}^*$ as a function of the network topology. Equity holdings impact risk-taking twice: (i) through the shareholding matrix via $\mathbf{C}$, and (ii) through the accounting balance via $(\mathbf{P}^T - \mathbf{P})\mathbf{1}$. This second effect arises from differences in the resources that can be allocated toward the risky asset when investments in equities by institutions, $e + d + \sum_{j\in\mathcal{I}} p_{ji} - \sum_{j\in\mathcal{I}} p_{ij}$, are not balanced. The entry i of vector $(\mathbf{P}^T - \mathbf{P})\mathbf{1}$ reflects the difference between the investment of other institutions in institution i's equity and the investment of institution i in other institutions' equities. It is useful to decompose risk-taking levels into $\mathbf{z}^* = \mathbf{z}^*_{\mathbf{RS}} + \mathbf{z}^*_{\mathbf{RE}}$ (where "RS" stands for the risk-sharing effect and "RE" stands for the resource effect):

$$\begin{cases} \mathbf{z}^*_{\mathbf{RS}} = \frac{e-(\rho-1)d}{(1-\ell_\beta)}(\mathbf{I} - \varepsilon\mathbf{C})^{-1}(\mathbf{I} + \mathbf{C})\mathbf{1} \\ \mathbf{z}^*_{\mathbf{RE}} = \frac{1}{(1-\ell_\beta)}(\mathbf{I} - \varepsilon\mathbf{C})^{-1}(\mathbf{I} + \mathbf{C})(\mathbf{P}^T - \mathbf{P})\mathbf{1} \end{cases}$$

Bow-tie networks. To complement the previous discussion on specific network structures to directed networks, bow-tie cross-shareholding networks have been identified in the empirical literature on industrial and finance economics (see Galeotti and Ghiglino (2021) and references therein; for a typical example, see the seven-institution network in Galeotti and Ghiglino (2021) in figure 3 therein).[31] They are particularly interesting to illustrate the powerful role of resource effects in shaping risks. Consider the following network shown in Fig. D.1: The in-section institution (institution 1) benefits from risk-

31. These networks have three classes of institutions: in-section, core, and out-section. In-section institutions invest in core institutions, core institutions invest among themselves and in out-section institutions, and out-section institutions do not invest in other institutions.

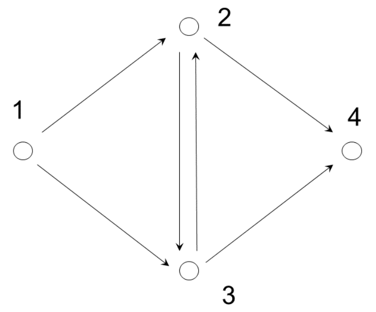

FIGURE D.1. A bow-tie network with four institutions.

sharing from the core institutions (institutions 2 and 3), but suffers from a negative resource effect (in that the sum of received investments is lower than the sum of investment in other institutions); core institutions benefit from each other only and have a null resource effect; the out-section institution (institution 4) benefits from no risk-sharing effect, but has a positive resource effect. Consider $\rho = 1.01$, $d = 100$, $e = 10$, $r = 1.02$, and $p = 5$. Then, for $l = 0.9$, $\mathbf{z}^* \simeq (113, 193, 193, 190)$; and for $l = 0.8$, $\mathbf{z}^* \simeq (48, 91, 91, 95)$. In this example, the out-section institution takes more risk than the in-section institution in both parameter sets. Furthermore, for a sufficiently high shock magnitude, the out-section institution also takes more risk than the core institutions. This example illustrates that the resource effect can dominate the risk-sharing effect.

Statics on integration. When the cross-shareholding network is undirected, there is no resource effect and Proposition 2 implies that integration increases optimal risk-taking. However, this result does not extend to the directed network case, that is when shareholding links are not reciprocated, nor when the amount invested in other institutions varies across institutions. We present an example where increased integration can decrease the contribution of resource effects to total optimal risk-taking ($\mathbf{1}^T\mathbf{z}^*_{RE}$) in directed networks. Consider indeed the following cross-shareholding network:

$$\mathbf{P} = \begin{pmatrix} 0 & p/3 & p/3 & p/3 \\ 0 & 0 & 0 & 0 \\ p & 0 & 0 & 0 \\ p & 0 & 0 & 0 \end{pmatrix}$$

Then, the resource effect is given by $p\boldsymbol{\gamma}$ where $\boldsymbol{\gamma} = (\mathbf{G}^T - \mathbf{G})\mathbf{1}$. To simplify, consider $\varepsilon = 0$, so that $\mathbf{z}^* = (\mathbf{I} + \mathbf{C})\left(\frac{e-(\rho-1)d}{1-\ell_\beta}\mathbf{1} + \frac{p}{1-\ell_\beta}\boldsymbol{\gamma}\right)$. The effect of p on the total contribution of resource effect to optimal risk-taking is then captured by: $p/(1-\ell_\beta)\mathbf{1}^T\mathbf{C}\boldsymbol{\gamma}$, and can be decreased when the level of integration of the financial network is increased. Recall here that matrix $\mathbf{C}$ depends on parameter p. Denoting $\mathbf{C}_p$ the value of this matrix under parameter p, we have: $\mathbf{1}^T\mathbf{C}_1\boldsymbol{\gamma} \sim -0.0328$ and $2\mathbf{1}^T\mathbf{C}_2\boldsymbol{\gamma} \sim -0.0789$. Therefore, the contribution of resource effects to total optimal risk-taking is here negative and decreasing with p. This comes from the negative correlation between the vector of out-degrees $\delta^O = (3,1,1,0)^T$ and the vector of resource effects $\boldsymbol{\gamma} = (-1,0,0,1)^T$.

To further explore the effect of integration, and to assess whether this (possibly negative) resource effect can outweigh the positive risk-sharing effect, we rely on simulations on undirected random graphs. Following Elliott et al. (2014), we first generate random graphs through the Erdös-Renyi procedure: for a fixed average degree $\delta \in \{1, \cdots, n-1\}$, each link is created with a probability of $\delta/(n-1)$. We alternatively generate random networks with power law degree distributions, using a Barabasi-Albert like procedure (that follows a preferential attachment mechanism), as follows:

1. Node 2 is attached to node 1 with probability 1. This gives $\mathbf{G}_2$ as the empty matrix plus the bilateral link 21 ($g_{12} = g_{21} = 1$).

2. For node $t = 3, 4, \cdots, n$, the probability of being linked with nodes $j = 1, 2, \cdots, t-1$ is equal to

$$\mathbb{P}_{tj}(\tau) = \frac{\tau}{n-1} \cdot \frac{\delta_j^{(t-1)}}{\sum_{k=1}^{t-1} \delta_k^{(t-1)}} \tag{D.4}$$

where degrees are defined over the network $\mathbf{G}_{t-1}$ (i.e. $\delta^{(t-1)} = \mathbf{G}_{t-1}\mathbf{1}$) defined as $\mathbf{G}_t = \mathbf{G}_{t-1}$ plus the set of links created at period t. Parameter $\tau > 0$ controls for the average density of the network. To model directed networks, we randomly draw the direction on the link, once created.

In the following, we present the results from these two procedures. We calibrate our simulations with $n = 20$, $\rho = 1.01$, $\underline{r} = 1.01$, $\ell = 0.85$, $d = 1500$ and $e = 100$. Figure D.2 depicts the effect of integration on average optimal risk-taking. We focus on $\mathbf{P}$ such that $\mathbf{P} = p\mathbf{G}$, with $\mathbf{G}$ a binary network satisfying $\mathbf{G}^T = \mathbf{G}$; we define $(\delta_i)_{i\in\mathcal{I}} = \mathbf{G1}$ as institution i's degree. With the above set of parameters, and the two forms of random network, we draw $1,000$ networks for each value of p in the fixed participation case ($p_{ij} = p \cdot g_{ij}$), and plot the average optimal risk-taking among banks and among runs. Figure D.2 presents the results for an average degree $\delta = 5$ under the Erdös-Renyi procedure and an average density $\tau = \delta \cdot n = 100$ under the Barabasi-Albert procedure.[32] Figure D.2 highlights several features of our model. First, it illustrates that the cross-shareholding network has significant effects on average risk-taking. In our parameter set, average z_i^* can increase by more than 50% with respect to that of isolated banks. Second, it shows that the positive risk-sharing effect of integration dominates on average the resource effect for random graphs, leading to an increase in average risk-taking. Third, it confirms that heterogeneity in network positions (which is higher under the Barabasi-Albert procedure) tends to decrease average risk-taking through resource effects, and dampens the effect

32. The average number of links is then the same under both procedures.

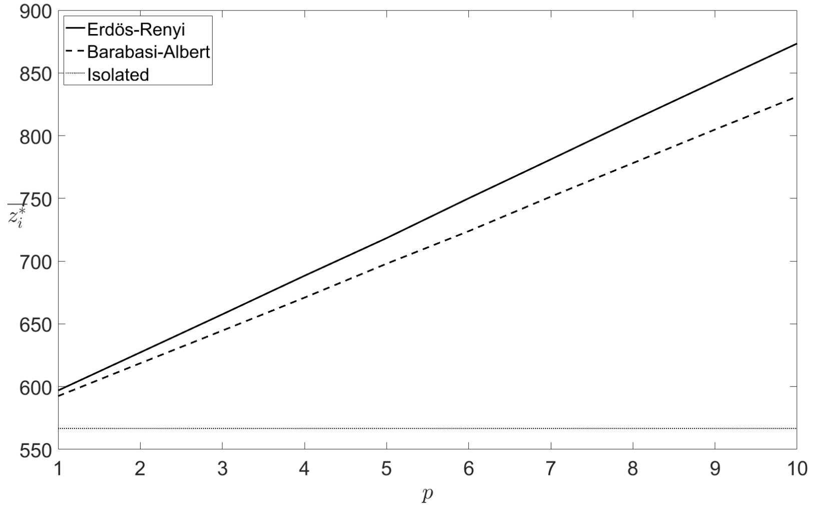

FIGURE D.2. The effect of integration on average optimal risk-taking

of integration. Overall, Figure D.2 suggests that integration has a monotonic impact on optimal risk-taking.[33]

33. Elliott et al. (2014) find (different) non-linear effects of integration and diversification on contagion.